جامعة وارث الأنبياء
كلية الهندسة

Mathematics

محاضرات في الرياضيات
كلية الهندسة
المرحلة الأولى.

اعداد

الأستاذ الدكتور زهاد عبد الجليل الموسوي

٢٠١٧ / ٢٠١٨

Contents

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د نهاد عبدالجليل

chapter 0

chapter 1

chapter 2

Chapter 3

Chapter 4

0.1 Inequalities

محاضرات في رياضيات
كلية الهندسة / جامعة بغداد
أ.د. نزار عبد الجليل مجيد

Rules for Inequalities

If a, b and c are real numbers, then

1. $a<b \Rightarrow a+c<b+c$
2. $a<b \Rightarrow a-c<b-c$
3. $a<b$ and $c>0 \Rightarrow ac<bc$
4. $a<b$ and $c<0 \Rightarrow bc<ac$, special case $-b<-a$
5. $a>0$ and $\frac{1}{a}>0$
6. If a and b are both positive or both negative then $a<b \Rightarrow \frac{1}{b}<\frac{1}{a}$

Ex1 Solve the following inequalities, and show their solution sets on the real line

$$-\frac{x}{3}<2x+1$$

Sol $-x<6x+3 \Rightarrow -7x<3 \Rightarrow x>\frac{-3}{7}$

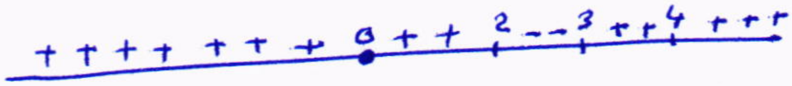
Ex2: Solve for x: $\frac{2x-5}{x-2} \leq 1$

Sol

$$\frac{2x-5}{x-2}-1 \leq 0$$

$$\frac{(2x-5)-(x-2)}{(x-2)} \leq 0$$

$$\frac{x-3}{x-2} \leq 0$$



$\therefore$ Set of sol $=(2,3]$

Functions, Domain and Range:

Def: A function f from a set D (Domain) to a set Y (Range) is any rule that assigns a unique

element $f(x) \in Y$ to each element $x \in D$

Ex: Find the domain and the range of the following Function

$$2-\sqrt{x} \geq 0$$

$$2 \geq \sqrt{x}$$

$$4 \geq x \Rightarrow \quad 0 \leq x \leq 4$$

D_f : $0 \leq x \leq 4$

$$\left.\begin{array}{l} \text{if } x=0 \Rightarrow y=\sqrt{2} \\ \text{if } x=4 \Rightarrow y=0 \end{array}\right\} \quad R_f : 0 \leq y \leq \sqrt{2}$$

Note Any polynomial has the domain R, For ex

$$f(x) = \frac{1}{2}x^3 + 3x^2 - x + \pi$$

Intervals:

1 The open interval. is the set of all real numbers that be strictly between two fixed numbers a and b

$$(a,b) \equiv a < x < b$$

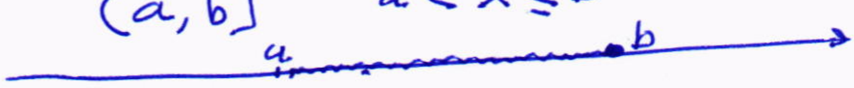
2 The closed interval is the set of all real numbers that contain both endpoints.

$$[a,b] \equiv a \leq x \leq b$$

3 Half open interval: is the set of all real numbers that contain one endpoint but not both

$$[a,b) \equiv a \leq x < b$$

$$(a,b] \quad a < x \leq b$$



Absolute value:

The absolute value of a number x, denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Absolute value properties..

1: $|-a| = |a|$

2: $|a.b| = |a|.|b|$

3: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

4: $|a+b| \leq |a| + |b|$

Ex:1 Solve the inquality involving absolute value

$$\left|5 - \frac{2}{x}\right| < 1$$

Sol

$$\left|5 - \frac{2}{x}\right| < 1$$

$$\Rightarrow -1 < 5 - \frac{2}{x} < 1$$

$$\Rightarrow -6 < \frac{-2}{x} < -4$$

$$\Rightarrow 3 > \frac{1}{x} > 2$$

$$\Rightarrow \frac{1}{3} < x < \frac{1}{2}.$$

Ex2 Solve:

$$|2x-3| \geq 1$$

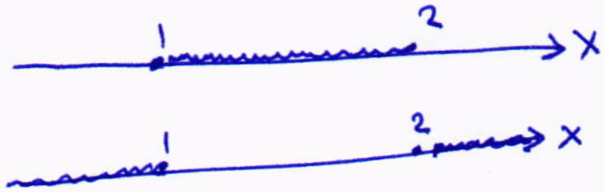
Sol $2x-3 \geq 1$ or $2x-3 \leq -1$

$x - \frac{3}{2} \geq \frac{1}{2}$ or $x - \frac{3}{2} \leq -\frac{1}{2}$

$x \geq 2$ or $x \leq 1$

The solution set $(-\infty, 1] \cup [2, \infty)$

محاضرات الرياضيات
كلية الهندسة
من اعداد أ.د. زياد عبد الجليل



Composition of functions:

Suppose that the outputs of a function f can be used as inputs of a function g, we can then hook f and g together to form a new function whose inputs are the inputs of f and whose outputs are the numbers.

$$(g \circ f)_x = g(f(x)).$$

Ex: Let $f(x) = \frac{x}{x-1}$ and $g(x) = 1 + \frac{1}{x}$ Find $(g \circ f)_x$ and $(f \circ g)x$.

Sol $(g \circ f)_x = g(f(x)) = g(\frac{x}{x-1}) = 1 + \frac{1}{\frac{x}{x-1}} = \frac{2x-1}{x}$

$(f \circ g)_x = f(g(x)) = f(1 + \frac{1}{x}) = \frac{1 + \frac{1}{x}}{1 + \frac{1}{x} - 1} = x + 1$

Ex2 Let $(g \circ f)_x = x$ and $f(x) = \frac{1}{x}$, Find $g(x)$

Sol $(g \circ f)_x = g(\frac{1}{x}) = x \Rightarrow g(x) = \frac{1}{x}$.

Increasing and decreasing Function:

Def. Let f be a function defined on a interval I, and let x_1, and x_2 be any two points in f.

1 if $f(x_2) > f(x_1)$, whenever $x_1 < x_2$, then f is said to be increasing on I

2 if $f(x_2) < f(x_1)$ whenever $x_2 < x_1$, then f is said to be decreasing on I.

Even Function and odd Function:

The graphs of even and odd functions have characteristic symmetry properties.

Def. A function $f(x)$ is an
Even function of x iff $f(-x) = f(x)$
Odd " " " iff $f(-x) = -f(x)$

Note The graph of an odd function is symmetric about the origin, since $f(-x) = -f(x)$

Ex. Even function: $f(x) = x^2$ is even function symmetry about y-axis.

Ex. odd function, $f(x) = x+1$, Not odd $f(-x) = -x+1$ but $-f(x) = -x-1$, The two are not equal.

Types of Functions:-

1. Linear Function:
$f(x) = mx + b$.

2. power function
$f(x) = x^a$, a is a constant

3. polynomials:
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

4. Rational Function
$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0.$$

Ex:
$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

5. Algebraic Function.
$$y = \frac{3}{4}(x^2 - 1)^{2/3}$$

6. Trigonometric function:
$f(x) = \sin x$, $f(x) = \cos x$

7. Exponential function.
$f(x) = a^x$. have domain $(-\infty, \infty)$, Range $(0, \infty)$

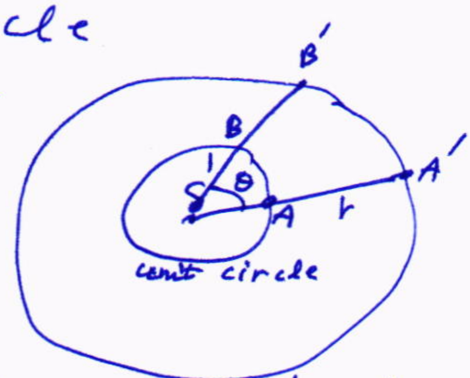
8 Logarithmic Function:

$f(x) = \log_a x$, $a \neq 1$, $a > 0$

Trigonometric Function:

The radian measure of the Angle at the center of the unit circle as shown in the Fig equals the length of the arc that ACB cuts from the unit circle

Thus $S = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians



circle of radius r

$$\frac{S}{r} = \frac{\theta}{1} \Rightarrow S = r \cdot \theta$$

Note if The circle is a unit circle having radius = 1 The one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180^\circ$$
$$2\pi \text{ radians} = 360^\circ$$

$$1 \text{ degree} = \frac{\pi}{180}$$

degrees to radians multiply by $\frac{\pi}{180}$

$$1 \text{ radians} = \frac{180}{\pi}$$

Radians to degree, multiply by $\frac{180}{\pi}$

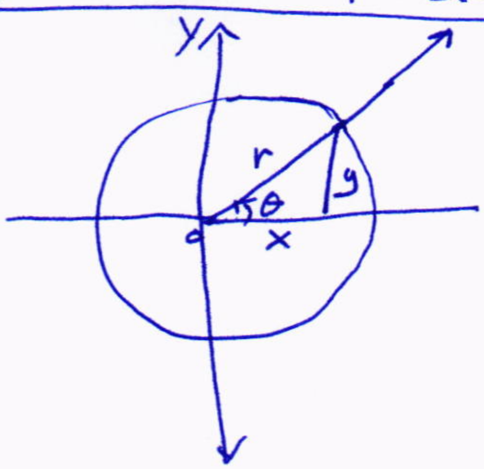
For ex 45° is radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4},$$

and $\pi/6$ radians is $\frac{\pi}{6} \cdot \frac{180}{\pi} = 30^\circ$

The Six Basic Trigonometric Functions

محاضرات الرياضيات
كلية الهندسة / المرحلة الاولى
من اعداد
أ.د. نهاد عبداللطيف



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

$\text{cota}\,\theta = \frac{x}{y}$

$\sec\theta = \frac{r}{x}$

$\text{cosec} = \frac{r}{y}.$

$\tan\theta = \frac{1}{\cot\theta}$, $\cos\theta = \frac{1}{\sec\theta}$, $\sin\theta = \frac{1}{\text{cosec}\,\theta}.$

$x = r\cos\theta$, $y = r\sin\theta.$

$\cos^2\theta + \sin^2\theta = 1$

Addition and Subtract Formulas:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

$\sin(\theta \pm 2\pi) = \sin\theta.$

$\cos(\theta \pm 2\pi) = \cos\theta$

$\sec(\theta \pm 2\pi) = \sec\theta$

$\tan(\theta \pm 2\pi) = \tan\theta$

$\cot(\theta \pm 2\pi) = \cot\theta$

$\csc(\theta \pm 2\pi) = \csc\theta$

$$\left.\begin{array}{l} \cos(-x) = \cos x \\ \sec(-x) = \sec x \end{array}\right\} \text{even}$$

$$\left.\begin{array}{l} \sin(-x) = -\sin x \\ \tan(-x) = -\tan x \\ \csc(-x) = -\csc x \\ \cot(-x) = -\cot x \end{array}\right\} \text{odd}$$

$$\sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos(x)$$

$$\cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin(x)$$

محاضرات في الرياضيات
كلية الهندسة
م. أ.د نزار عبد الجليل

Limits

if $f(x)$ is approaches to L, when x approaches to x_0 from left and right then we say that the limit exist at $x = x_0$

or $\lim_{x \to x_0} f(x) = L$

if $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$.

<u>Theorem:</u> if $\lim_{x \to a} f(x) = L_1$, $\lim_{x \to a} g(x) = L_2$, and k is constant then:

1) $\lim_{x \to a} \left(f(x) \pm g(x) \right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L_1 \pm L_2$

2) $\lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x) = k \cdot L_1$

3) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L_1}{L_2}$, $L_2 \neq 0$

4) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, $L_1 \geq 0$

<u>Ex</u> Find $\lim_{x \to 5} (x^2 - 4x + 3) = 5^2 - 4 \times 5 + 3 = 8$

<u>Ex</u> Find $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$= \lim_{x \to 2} (x+2) = 4$

<u>Ex.</u> $\lim_{x \to 4} \dfrac{2 - x}{(x-4)(x+2)}$

<u>Sol</u> $\lim_{x \to 4^-} \dfrac{2 - x}{(x-4)(x+2)} = -\infty$

$\lim_{x \to 4^+} \dfrac{2 - x}{(x-4)(x+2)} = +\infty$

does not exist

H.W Find

1. $\lim_{x \to -\infty} \dfrac{4x^2 - x}{2x^3 - 5}$ Ans : 0

2. $\lim_{x \to +\infty} \dfrac{3x+5}{6x+8}$ Ans : $\frac{1}{2}$

3. $\lim_{x \to -\infty} \sqrt[3]{\dfrac{3x+5}{6x-8}}$ Ans : $\sqrt[3]{\frac{1}{2}}$

4. $\lim_{x \to -\infty} \dfrac{\sqrt{x^2-2}}{3x-6}$ Ans : $-\frac{1}{3}$

5. $\lim_{x \to 3} \quad f(x) = \begin{cases} x^2 - 5 & , \quad x \le 3 \\ \sqrt{x+13} & , \quad x > 3 \end{cases}$ Ans : 4

Ex: Find $\lim_{x \to 0} \dfrac{x}{|x|}$

$$= \lim_{x \to 0} \frac{x}{|x|} = \begin{cases} \frac{x}{x} & , \quad x \ge 0 \\ \frac{x}{-x} & , \quad x < 0 \end{cases} = \begin{cases} 1 & , \quad x \ge 0 \\ -1 & , \quad x < 0. \end{cases}$$

The lim does not exist

Ex: Find $\lim_{x \to 1} \dfrac{x^3 + x^2 - 5x + 3}{x^3 - 3x + 2}$ Ans: $\frac{4}{3}$

Theorem

1. $\lim_{x \to 0} \dfrac{\sin ax}{ax} = 1$, $a \ne 0$
2. $\lim_{x \to 0} \sin x = 0$
3. $\lim_{x \to 0} \cos x = 1$
4. $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$
5. $\lim_{x \to \infty} \dfrac{\sin x}{x} = 0$
6. $\lim_{x \to \infty} \dfrac{\cos x}{x} = 0$

Ex $\lim_{x\to\infty} \sin\left(\frac{\pi x}{2-3x}\right)$

$= \sin \lim_{x\to\infty} \frac{\pi x}{2-3x}$

$= \sin \lim_{x\to\infty} \left(\frac{\pi}{\frac{2}{x}-3}\right) = \sin\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

Ex.: $\lim_{x\to\pi} \frac{(\pi-x)}{\sin x} = \lim_{x\to\pi} \frac{(\pi-x)}{\sin(\pi-x)}$

Let $\pi - x = t \Rightarrow x = \pi - t$, if $x\to\pi$ then $t=0$

$= \lim_{t\to 0} \frac{t}{\sin t} = \lim_{t\to 0} \frac{t/t}{\frac{\sin t}{t}} = 1$

Ex: $\lim_{x\to\infty} \frac{2x + x\sin x}{5x^2-2x+1}$

$= \lim_{x\to\infty} \frac{\frac{2x}{5x^2} + \frac{x\sin x}{x^2}}{\frac{5x^2}{x^2} - \frac{2x}{x^2} + \frac{1}{x^2}} = \frac{\lim_{x\to\infty}\frac{2}{x} + \lim \frac{\sin x}{x}}{\lim_{x\to\infty} 5 - \lim_{x\to\infty}\frac{2}{x} + \lim_{x\to\infty}\frac{1}{x^2}} = 0$

(Continuity)

Definition:

A function f is said to be continuous at a point c, if the following condition are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

if one or more conditions in this definition fails to hold, then f is called discontinuous at c, and c is called the point of discontinuity of f.

Ex Let $f(x) = \frac{x^2-4}{x-2}$

1. The function f at $x=2$ is undefined, so that f is discontinuous.

Ex Does the function $f(x) = \begin{cases} 2x+3, & x \le 4 \\ 7+\frac{16}{x}, & x > 4 \end{cases}$

continuous at $x=4$

1. $f(4) = 2 \times 4 + 3 = 11$ defined at $x=4$
2. $\lim_{x \to 4^+} f(x) = \lim_{x \to 4} \left(7 + \frac{16}{x}\right) = 11$ exist.

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (2x+3) = 11$

3. $\lim_{x \to 4} f(x) = f(4) = 1$

Theorem: If the functions f and g are continuous at c, then:

1. $f+g$ is continuous at c.
2. $f-g$ " " " c
3. $f \cdot g$ " " " c
4. f/g " " " c, if $g(c)$ is discontinuous at c, if $g(c)=0$.

Ex where is $h(x) = \dfrac{x^2-9}{x^2-5x+6}$ continuous

Sol $x^2-5x+6 = (x-3)(x-2)$

$x=2$, $x=3$

Hence $h(x)$ is continuous everywhere expect at these points $(2, 3)$

Theorem: If f and g are continuous at c, then $f \circ g$ is continuous at c.

محاضرات في الرياضيات
كلية الهندسة / المرحلة الاولى
مدرس المادة أ.د نهاد عبد الجليل

H.W

1. Find $\lim_{x\to c} \dfrac{x^4+x^2-1}{x^2+5}$, Ans $\dfrac{c^4+c^2-1}{c^2+5}$

2. $\lim_{x\to -2} \sqrt{4x^2-3}$, Ans: $\sqrt{13}$

3. $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-x}$, Ans: 3

4. suppose $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$, if $\lim_{x\to c} f(x)=5$, $\lim_{x\to c} g(x)=-2$

Find $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$, Ans $\dfrac{5}{7}$

Chapter 1
Differentiation

Definition: Let $y=f(x)$ be a function, then the derivative of f with respect to x is defined by $\dfrac{dy}{dx}=y'=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Theorem: If f and g are diffentiable function at x and k is constant then.

1. $\dfrac{d(k)}{dx}=0$
2. $\dfrac{d}{dx}(f(x)\pm g(x))=\dfrac{d}{dx}(f(x)\pm\dfrac{d}{dx}g(x))$
3. $\dfrac{d}{dx}(f(x)\cdot g(x))=f(x)\cdot\dfrac{d}{dx}g(x)+g(x)\cdot\dfrac{d}{dx}f(x)$
4. $d(k\,f(x))=k\dfrac{d}{dx}f(x)$
5. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{g(x)\cdot\frac{d}{dx}f(x)-f(x)\cdot\frac{d}{dx}g(x)}{(g(x))^2}$, provided $g(x)\neq 0$
6. $\dfrac{d}{dx}x^n=n\,x^{n-1}$

Ex: Let $y = \left(\frac{x-1}{x+1}\right)(2x^7 - x^2)$, Find y'

Sol $y' = \left(\frac{x-1}{x+1}\right)(14x^6 - 2x) + (2x^7 - x^2)\frac{(x+1)\cdot 1 - x(x-1)\cdot 1}{(x+1)^2}$

$= \left(\frac{x-1}{x+1}\right)(14x^6 - 2x) + \frac{2(2x^7 - x^2)}{(x+1)^2}$

Ex2: Find y'', where $y = 6x^5 - 4x^2$

Sol $y' = 30x^4 - 8x$

$y'' = 120x^3 - 8$

$y''_{at\ 1} = 112$

Derivatives of Trigonometric Function:

1. $y = \sin x \Rightarrow y' = \cos x$

2. $y = \cos x \Rightarrow y' = -\sin x$

3. $y = \tan x \Rightarrow y' = \sec^2 x$

4. $y = \cot x \Rightarrow y' = -\csc^2 x$

5. $y = \sec x \Rightarrow y' = \sec x \tan x$

6. $y = \csc x \Rightarrow y' = -\csc x \cot x$

Chain Rule:

1. if $y = f(u)$, $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Ex. Let $y = u^{10}$, and $u = x^3 + 7x + 1$, Find y'.

Sol: $\frac{dy}{du} = 10u^9$, $\frac{du}{dx} = 3x^2 + 7 \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx}=(10u^9)\cdot(3x^2+7)$$

$$\frac{dy}{dx}=[10(x^3+7x+1)^9][3x^2+7]$$

طريقة ثانية

$$y=(x^3+7x+1)^{10} \Rightarrow \frac{dy}{dx}=[10(x^3+7x+1)^9]\cdot[3x^2+7]$$

Implicit Differentiation:

Ex: Find y', if $y^2=x$

Sol $y^2=x$

$2y\cdot y'=1$

$y'=\frac{1}{2y}$

Ex: Find y', if $2x^3-3y^2=8$

Sol $6x^2-6yy'=0$

$6yy'=6x^2$

$yy'=x^2 \Rightarrow y'=\frac{x^2}{y}$, when $y\neq 0$.

$3y^2=2x^3-8 \Rightarrow y^2=\frac{2x^3-8}{3}$, $y=\mp\sqrt{\frac{2x^3-8}{3}}$

$$\therefore y'=\frac{x^2}{\mp\sqrt{\frac{2x^3-8}{3}}}$$

H.W

1 Find y' if $y^2=x^2+\sin xy$, Ans: $y'=\frac{2x+y\cos xy}{2y-x\cos xy}$

2 Find y', if $x^3+y^3-9xy=0$, Ans: $y'=\frac{3y-x^2}{y^2-3x}$.

3 Find y', if $y=x^2y+xy^2=6$, Ans: $\frac{-2xy-y^2}{x^2+2xy}$

4: Find y', if $x=\tan y$, Ans. $\cos^2 y$.

محاضرات في الرياضيات
كلية الهندسة / المرحلة الاولى
مدرس المادة أ.د. زياد عبد الجليل

Application of Differentiation

Theorem:

1. if $f'(x) > 0$ on (a,b), then f is increase on (a,b).
2. if $f'(x) < 0$ on (a,b) then f is decrease on (a,b)

Ex: On which intervals is the function:
$f(x) = x^3 - 3x^2 + 1$, increase or decrease.

Sol

$f'(x) = 3x^2 - 6x$
$= 3x(x-2) = 0.$

$x = 0$ or $x = 2$

$+ + + + +$ (0) $- - - -$ (2) $+ + + +$ Sign of y'

So $f(x)$ is increase on $(-\infty, 0) \cup (2, \infty)$
and decrease on $(0, 2)$.

Def. A function $f(x)$ is concave up on (a,b)
if $y'' > 0$
and concave down on (a,b)
if $y'' < 0$.

Ex: Find the interval where the function
$f(x) = x^3 - 3x^2 + 1$ concave up, concave down

Sol

$f'(x) = 3x^2 - 6x$
$f''(x) = 6x - 6 \Rightarrow x = 1$

$- - - - -$ (1) $+ + + + + + +$ Sign of $f''(x)$

f is concave up on $(1, \infty)$, and concave down on $(-\infty, 1)$

Def: let f be a continuous function on $[a,b]$, and f

changes direction of concavity at x_0 then $(x_0, f(x_0))$ called inflection point of f.

or $(x_0, f(x_0))$ is an inflection point if $y''(x)=0$.

Ex: Find the Location of all inflection points of:

$$f(x) = x^4 - 8x^2 + 16$$

$$f'(x) = 4x^3 - 16x$$

$$f''(x) = 12x^2 - 16 \Rightarrow 4(3x^2 - 4) = 0$$

$$x = \pm \frac{2}{\sqrt{3}}$$

$\left(\frac{2}{\sqrt{3}}, 7.1\right)$ and $\left(\frac{-2}{\sqrt{3}}, 7.1\right)$ are inflection points.

Ex: Find the intervals in which the function:

$$f(x) = x^{1/3}(x+4)$$

Is increasing, decreasing, concave up, concave down and inflection point.

Sol
$$f(x) = x^{1/3}(x+4)$$

$$f'(x) = y' = x^{1/3} \cdot 1 + (x+4)\left(\frac{1}{3}\right)x^{-2/3}$$

$$f'(x) = x^{1/3} + \frac{x+4}{3x^{2/3}}$$

$$= \frac{4x+4}{3x^{2/3}}$$

- - - - - - - (−1) + + + (0) + + + + + + + Sign of y'

f increase on $(-1, 0)$, $(0, \infty)$

f decrease on $(-\infty, -1)$

$$f''(x) = \frac{4}{3}(x+1)\left(\frac{-2}{3}x^{-5/3}\right) + x^{-2/3}$$

$$\frac{4(x-2)}{9\,x^{5/3}} = 0 \Rightarrow x = 2$$

$+ + + + + + + +$ (0) $- - - - -$ (2) $+ + + + + +$ Sign of y''

f concave up on $(-\infty, 0)$, $(2, \infty)$ concave down on $(0, 2)$, $(2, 7.6)$ is the inflection point.

H.W.:

1. On which intervals is the function $f(x) = x^3 - 3x^2 + 1$, increasing, decreasing

Ans: inc if $x<0$
dec if $0<x<2$
inc if $x>2$

2/ where is the function $f(x) = x^3 - 3x^2 + 1$, concave up. concave down?

Ans: f concave up if $x>1$
f " down if $x<1$

3. Determine open intervals on which f is
a) increasing, b) decreasing, c) concave up, d) concave down
e) Find the Location of all inflection points.
$$f(x) = x^2 - 5x + 6$$

Ans:

a) $(5/2, \infty)$, b $(-\infty, 5/2)$, c $(-\infty, \infty)$, d (none)
e (none)

4: Find the inflections points, if any, of the function $f(x) = (x-a)^4$ Ans: none

5/ prove that:
a) $f(x) = x^2$ is increasing on $(0, -\infty)$
b) $f(x) = x^2 - 2x$ is decreasing on $(-\infty, 1]$

محاضرات في الرياضيات
كلية الهندسة/المرحلة الاولى
من اعداد أ.د نهاد عبد الخليل

Sketching graphs

Ex: sketch the graph of the curve $f(x) = x^3 - 3x + 2$

Sol $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0$

$x = 1$ and $x = -1$

$+ + + + + + + + \quad -1 \quad - - - - - - \quad 1 \quad + + + + + + +$ → Sign of y'

f increase on $(-\infty, -1)$, $(1, \infty)$

f decrease on $(-1, 1)$

$f(-1) = 4 \Rightarrow (-1, 4)$ is relative maximum

$f(1) = 0 \Rightarrow (1, 0)$ is relative minimum

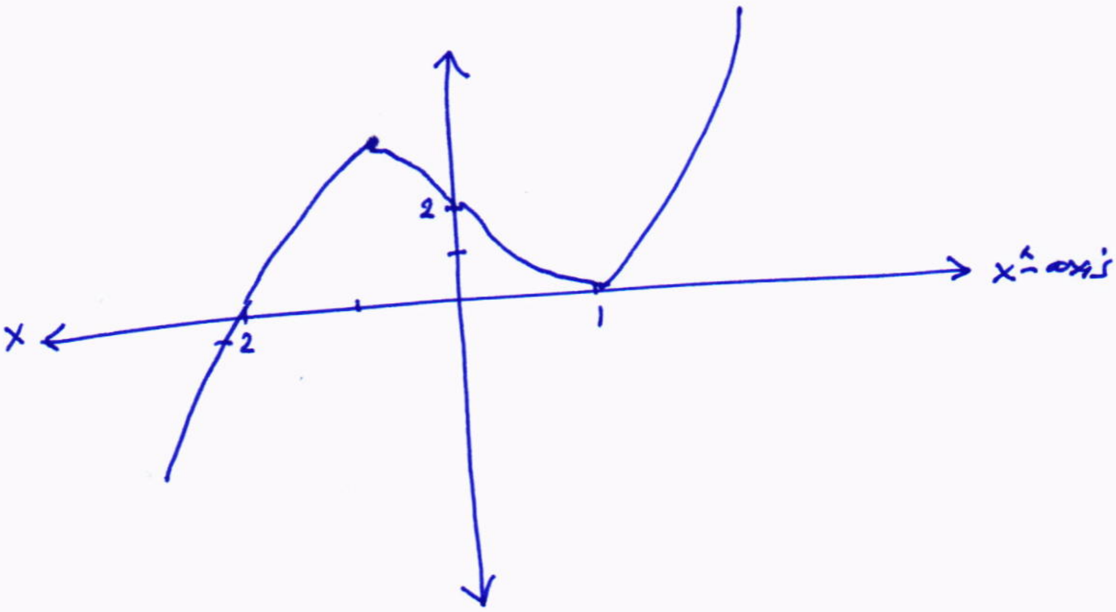
$f' \begin{cases} - \to + \\ + \to \max \end{cases}$

$f''(x) = 6x \Rightarrow \quad 6x = 0 \Rightarrow x = 0$

$(0, 2)$ is inflection point.

$- - - - - - \quad 0 \quad + + + + +$ → Sign of y''

f is concave up on $(0, \infty)$, and concave down on $(-\infty, 0)$



Exercise

a/ Find values of the constants a, b and c that will make

$f(x) = \cos x$, and $g(x) = a + bx + cx^2$

satisfy the conditions.

$f(0) = g(0)$, $f'(0) = g'(0)$, and $f''(0) = g''(0)$

Ans: $a = 1$, $b = 0$, $c = -\frac{1}{2}$

b. Find values for b and c that will make

$f(x) = \sin(x + a)$, and $g(x) = b \sin x + c \cos x$

satisfy the conditions

$f(0) = g(0)$ and $f'(0) = g'(0)$.

Ans. $b = \cos a$, $c = \sin a$

c/ For the determined values of a, b and c, what happens for the third and fourth derivatives of f and g in each parts a and b.

$\boxed{\text{Chapter 2}}$

2.1 Integration:

Def. $\frac{d}{dx}[f(x)] = F(x)$ then $\int F(x)\, dx = f(x) + c$

Def: if $y = f(x)$ is nonnegative and integrable over closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b

$$A = \int_a^b f(x)\, dx.$$

Properties of integration:

1. $\int c\ f(x)\,dx = c\int f(x)\,dx$

2. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c \quad , \quad n \neq -1$

4. $\int \cos u\,du = \sin u + c$

5. $\int \sin u\,du = -\cos u + c$

6. $\int \sec^2 u\,du = \tan u + c$

7. $\int \csc^2 u\,du = -\cot u + c$

8. $\int \sec u \tan u\,du = \sec u + c$

9. $\int \csc u \cot u\,du = -\csc u + c$

Examples:

Ex1 Evaluate $\int 4\cos x\,dx$

Sol. $\int 4\cos x\,dx = 4\int \cos x\,dx$

$= 4(\sin x + c)$

$= 4\sin x + 4c$

Ex2 $\int (x + x^2)\,dx$

Sol: $\int (x+x^2)\,dx = \int x\,dx + \int x^2\,dx$

$= \dfrac{x^2}{2} + \dfrac{x^3}{3} + c_1 + c_2$

$= \dfrac{x^2}{2} + \dfrac{x^3}{3} + c$

Ex3 $\int \dfrac{x^2 - 2x^4}{x^4}\,dx$

Sol: $\int \left(\dfrac{1}{x^2} - 2\right) dx \Rightarrow \int (x^{-2} - 2)\,dx = \dfrac{x^{-1}}{-1} - 2x + c$

Integration by Substitution:

Ex Evaluate

$$\int (x^2+1)^{50}\, 2x \cdot dx$$

Sol let $u = x^2+1$

$$\Rightarrow \frac{du}{dx} = 2x$$

$$\int (x^2+1)^{50} \cdot 2x \cdot dx = \int \left[u^{50} \frac{du}{dx}\right] \cdot dx = \int u^{50}\, du$$

$$= \frac{u^{51}}{51} + c = \frac{(x^2+1)^{51}}{51} + c.$$

Ex: Evaluate

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx$$

Sol: let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow du = \frac{1}{2\sqrt{x}}\, dx$

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx = \int \cos u\, du = \sin u + c = \sin\sqrt{x} + c$$

Ex Evaluate:

$$\int \sin(x+9)\, dx$$

Sol: let $u = x+9$

$$du = dx$$

$$\int \sin(x+9)\, dx \Rightarrow \int \sin u\, du = -\cos u + c$$

$$\Rightarrow -\cos(x+9) + c$$

Ex: Evaluate:

$$\int \cos 5x\, dx \Rightarrow \int (\cos u) \cdot \frac{1}{5}\, du = \frac{1}{5} \int \cos u\, du$$

$u = 5x \Rightarrow du = 5dx$

$$\int \cos 5x\, dx = \frac{1}{5} \sin u + c \Rightarrow \frac{1}{5} \sin 5x + c$$

The definite integral:

Ex: Evaluate

$$\int_0^3 (x^3-4x+1)\,dx$$

$$= \frac{x^4}{4} - 4\frac{x^2}{2} + x \Big]_0^3 = \left(\frac{81}{4} - 18 + 3\right) - 0 = \frac{21}{4}.$$

properties:

1) $$\int_a^a f(x)\,dx = 0$$

2) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\,dx$$

if $a<c<b$

Ex. $$\int_5^5 x^3\,dx$$

Sol $$\int_5^5 x^3\,dx = \frac{x^4}{4}\Big|_5^5 = \frac{625}{4} - \frac{625}{4} = 0.$$

Ex: If $\int_0^x \frac{\sin t}{t}\,dx$, Find $F'(x)$

$$F'(x) = \frac{\sin x}{x}$$

[Application of the definite integral]

1 Area between two curves:

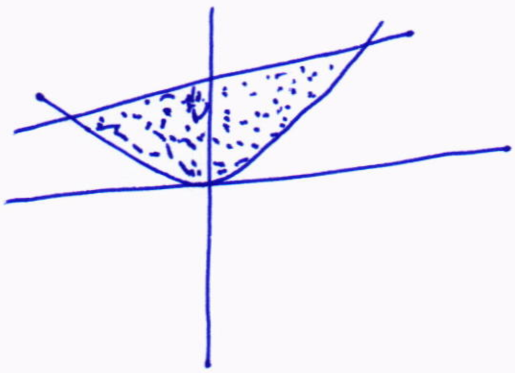
a if $f(x) \geq g(x)$, for $a \leq x \leq b$ then:

$$A = \int_a^b [f(x) - g(x)]\, dx$$

b if $f(y) \geq g(y)$, for $c \leq y \leq d$ then:

$$A = \int_c^d [(f(y) - g(y)]\, dy.$$

Ex: Find the area of the region enclosed between the curves $y_1 = x^2$ and $y_2 = x + 6$.



Sol

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$\therefore x = -2, \quad x = 3$$

check $x = 0$.

$y_1 = 0$ (below), $y_2 = 6$ (above)

$$A = \int_{-2}^{3} (y_2 - y_1)\, dx = \int_{-2}^{3} [(x+6) - x^2]\, dx$$

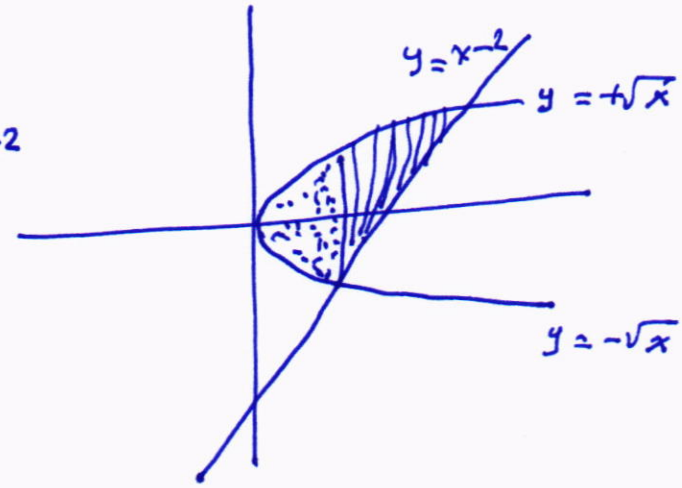
$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \left[\frac{3^2}{2} + 6 \times 3 - \frac{3^3}{3}\right) - \left(\frac{(-2)^2}{2} + 6 \cdot -2 - \frac{(-2)^3}{3}\right)$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6} \text{ units}^2$$

Ex Find the area of the region enclosed between the curves:

$x = y^2$ and $y = x - 2$



Method (1):

$y_1 = \sqrt{x}$, $y_2 = -\sqrt{x}$, $y_3 = x - 2$

$\sqrt{x} = -\sqrt{x}$
$\sqrt{x} + \sqrt{x} = 0$
$2\sqrt{x} = 0$
$\sqrt{x} = 0$

$x = 0$

$\pm\sqrt{x} = x - 2$
$x = (x-2)^2$
$x^2 - 5x + 4 = 0$
$(x-1)(x-4) = 0$
$\therefore x = 1$, $x = 4$

$A = A_1 + A_2$

$$A = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\, dx + \int_1^4 [\sqrt{x} - (x-2)]\, dx$$

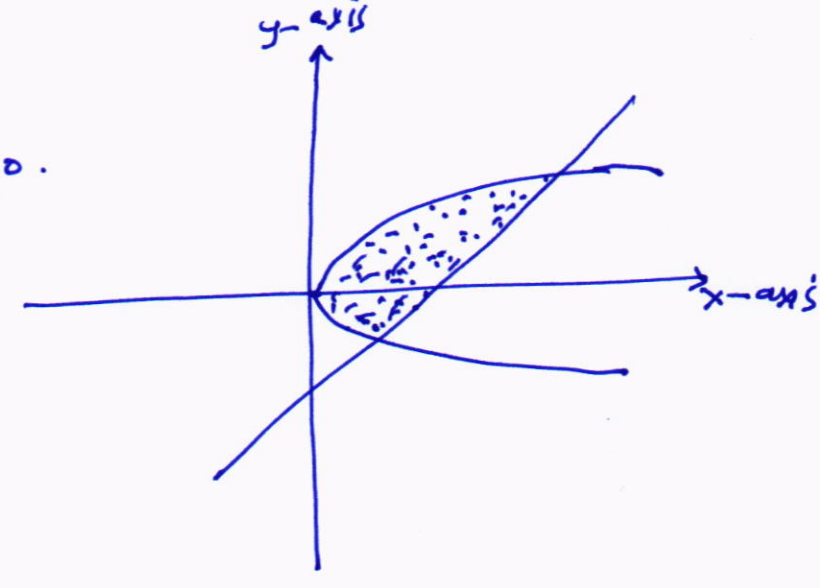
$$= \int_0^1 2\sqrt{x}\, dx + \int_1^4 [\sqrt{x} - x + 2]\, dx$$

$$= 2\int_0^1 x^{1/2}\, dx + \int_1^4 [x^{1/2} - x + 2]\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right]_1^4 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$

Method (2)

$y^2 = y + 2$, $y^2 - y - 2 = 0$.
$(y+1)(y-2) = 0$
$y = -1$, $y = 2$

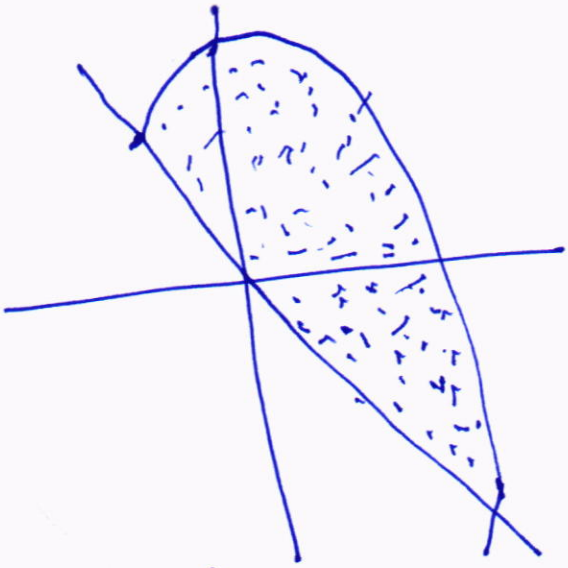


$$A = \int_{-1}^{2} [(y+2) - y^2]\, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{2^2}{2} - 2\times 2 - \frac{2^3}{3}\right] - \left[\frac{(-1)^2}{2} - 2(-1) - \frac{(-1)^3}{3}\right] = \frac{9}{2} \text{ units}^2$$

Ex: Find the area of the region enclosed by the Parabola $y = 2 - x^2$ and the line $y = -x$.



Sol

$$2 - x^2 = -x$$

$$x^2 - x - 2 = 0.$$

$$(x-2)(x+1) = 0$$

$$x = -1 \quad , \quad x = 2$$

$$A = \int_a^b [f(x) - g(x)]\, dx$$

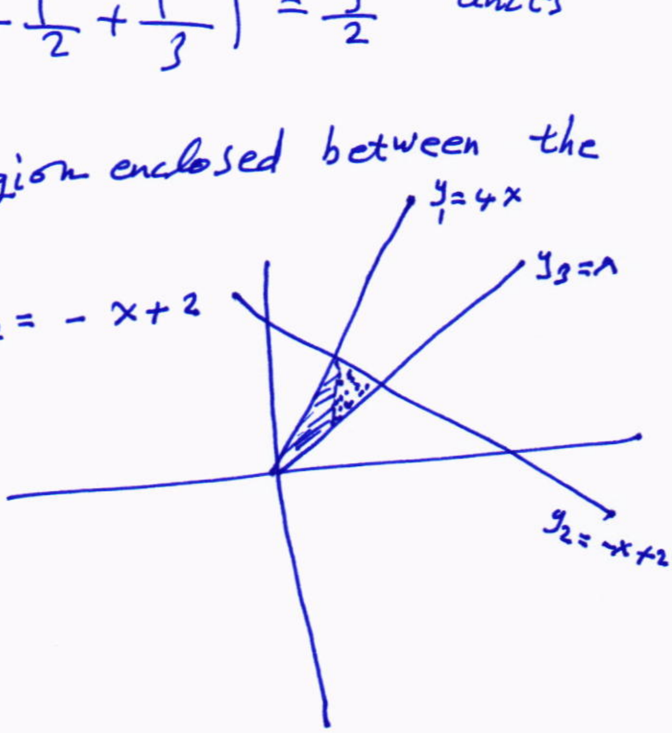
$$= \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2} \text{ units}^2$$

Ex: Find the area of the region enclosed between the curves.

$y_3 = x$, $y_1 = 4x$, $y_2 = -x + 2$



$$y_1 = y_2$$

$$4x = -x + 2$$

$$5x = 2 \Rightarrow x = \frac{2}{5}$$

$y_1 = y_2$

$4x = 0 \Rightarrow x = 0$

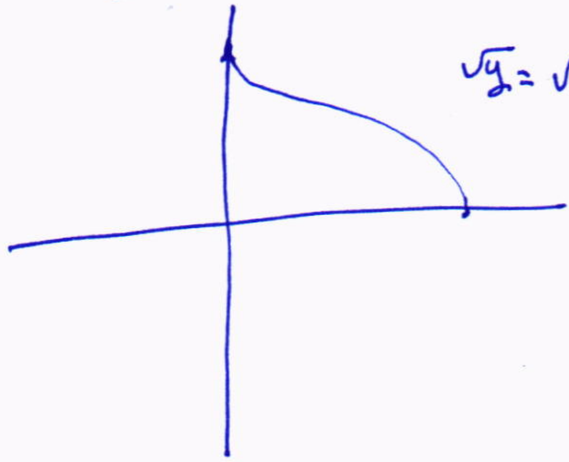
$A = A_1 + A_2$

$$A = \int_0^{2/5} (4x - x)\,dx + \int_{2/5}^{1} [(-x+2) - x]\,dx$$

$$= \left[4\frac{x^2}{2} - \frac{x^2}{2}\right]_0^{2/5} + \left[-\frac{x^2}{2} + 2x - \frac{x^2}{2}\right]_{2/5}^{1}$$

$$= \left[\frac{3x^2}{2}\right]_0^{2/5} + \left[-x^2 + 2x\right]_{2/5}^{1} = \frac{3}{5}$$

Ex: Find the area bounded by $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and the coordinate axes



$\sqrt{y} = \sqrt{a} - \sqrt{x}$

Sol $\sqrt{x} + \sqrt{y} = \sqrt{a}$

put $x = 0$

$\sqrt{y} = \sqrt{a}$

$y = a \Rightarrow (0, a)$

put $y = 0$

$\sqrt{x} = \sqrt{a}$

$x = a \Rightarrow (a, 0)$

$\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\sqrt{y} = \sqrt{a} - \sqrt{x}$

$y = (\sqrt{a} + \sqrt{x})^2 \Rightarrow y = a - 2\sqrt{a}\sqrt{x} + x$

$$\therefore A = \int_0^a (a - 2\sqrt{a}\sqrt{x} + x)\,dx = \left[ax - 2\sqrt{a}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2}\right]_0^a$$

$$= a^2 - \frac{4}{3} a^{\frac{1}{2}} a^{\frac{3}{2}} + \frac{a^2}{2}$$

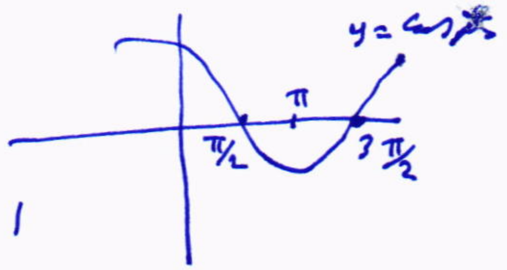
$$= a^2 - \frac{4}{3}a^2 + \frac{a^2}{2} = \frac{6a^2 - 8a^2 + 3a^2}{6} = \frac{a^2}{6} \text{ units}^2$$

H.W

1. show that the area under $y = x^3$ over the interval $[0, b]$ is $b^4/4$

2. Find the area under the curve $y = 9 - x^2$ over the interval $[0, 3]$ Ans 18.

3. Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$.



Ans : 1

4. Find the total area that is between the curve $y = x^2 - 3x - 10$, and the interval $[-3, 8]$.

Ans : $\frac{203}{2}$

5. Find the areas of the regions enclosed by the curves and lines

a/ $y = x$, $y = \frac{1}{x^2}$, $x = 2$, ans : 1

b/ $\sqrt{x} + \sqrt{y} = 1$, $x = 0$, $y = 0$, ans : $\frac{1}{6}$

c/ $x = 2y^2$, $x = 0$, $y = 3$, ans : 18

d/ $y = \sin x$, $y = x$, $0 \le x \le \frac{\pi}{4}$, ans : $\frac{\pi^2}{32} + \frac{\sqrt{2}}{2} - 1$

e/ $y^2 = 4x$, $y = 4x - 2$, ans : $\frac{9}{8}$

f/ $y = 2\sin x$, $y = \sin 2x$ $0 \le x \le \pi$ ans : 4

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د. نهاد عبد الجليل

VOLUMES

1. volumes by disks.
2. volumes by washers.
3. volumes by cylindrical shells:

1. volumes by disks.

$$V = \int_a^b \pi \left(f(x) \right)^2 dx .$$

Ex: Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$, and the lines $y=1$, $x=4$ about the line $y=1$.

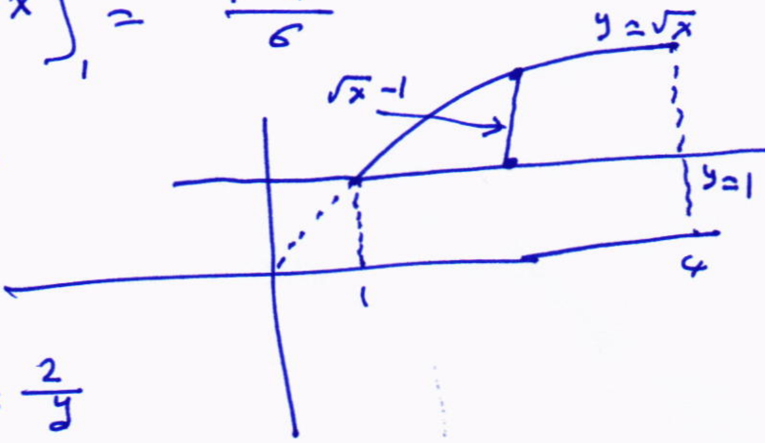
Sol

$$V = \int_a^b \pi \left(f(x) \right)^2 dx$$

$$V = \int_1^4 \pi \left[\sqrt{x} - 1 \right]^2 dx$$

$$V = \int_1^4 \left[x - 2\sqrt{x} + 1 \right] dx$$

$$V = \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}$$



Ex: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis.

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
م. اعداد أ.م.د. نهاد عبد الجليل

Sol $V = \int_a^b (f(x))^2 \, dx$

$$V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy.$$

$$= \pi \int_1^4 \frac{4}{y^2}\, dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] = 3\pi .$$

Ex: Find the volume of the solid generated by revolving the region between the parabola $x = y^2+1$ and the line $x=3$ about the line $x=3$

Sol $y = -\sqrt{2}$, $y = +\sqrt{2}$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[2 - y^2 \right]^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[4 - 4y^2 + y^4 \right] dy$$

$$= \pi \left[4y - \frac{4}{3} y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64 \pi \sqrt{2}}{15}$$

2 Volumes by washers:

$$V = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx$$

Ex: The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about the x-axis to generate a solid, Find the volume of the Solid.

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د. نهاد عبد الجليل

Sol $x^2+1=-x+3$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2\ ,\ x=1$

$$V=\int_a^b \pi\left[(f(x))^2-(g(x))^2\right]dx$$

$$V=\pi\int_{-2}^{1}(-x+3)^2-(x^2+1)^2\,dx$$

$$V=\pi\int_{-2}^{1}(8-6x-x^2-x^4)\,dx$$

$$V=\pi\left[8x-3x^2-\frac{x^3}{3}-\frac{x^5}{5}\right]_{-2}^{1}=\frac{117\pi}{5}$$

Ex: Find the volume of the solid, that is the region bounded by the parabola $y=x^2$ and the line $y=2x$, revolved about the y-axis

Sol $f(y)=\sqrt{y}\ ,\quad g(y)=\frac{y}{2}\ ,\ c=0\ ,\ d=4$

$$V=\int_c^d \pi\left[(f(y))^2-(g(y))^2\right]dy$$

$$=\int_0^4 \pi\left(\left[\sqrt{y}\right]^2-\left[\frac{y}{2}\right]^2\right)dy$$

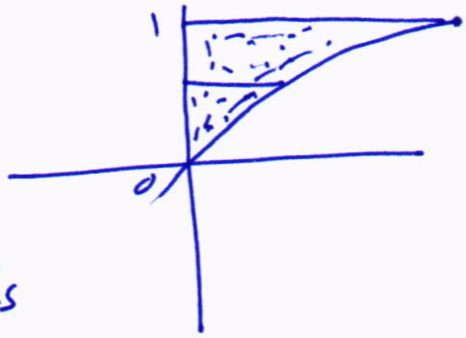
$$=\pi\int_0^4\left(y-\frac{y^2}{4}\right)dy=\pi\left[\frac{y^2}{2}-\frac{y^3}{12}\right]_0^4$$

$$=\frac{8}{3}\pi$$

H.W

1 Find the volume of the solid generated by revolving the shaded region about the y-axis

$x = \tan\left(\frac{\pi}{4}\right) y$ ans: $4 - \pi$



2 Find the volumes of the solids generated by revolving the regions bounded by the lines and the curves in the following

a/ $y = x^2$, $y = 0$, $x = 2$ ans: $\frac{32\pi}{5}$

b $y = \sqrt{\cos x}$, $0 \le x \le \frac{\pi}{2}$, $y = 0$, $x = 0$ Ans π

c The region between the curve $y = \frac{1}{2\sqrt{x}}$ and the x-axis from $x = \frac{1}{4}$ to $x = 4$ Ans $\frac{\pi}{2} \ln 4$

3 Find the volumes of the solids generated by revolving the regions bounded by the lines and the curves in the following.

a The region enclosed by $x = \sqrt{5}\, y^2$, $x = 0$, $y = -1$, $y = 1$

Ans 2π

b $x = 2/\sqrt{y+1}$, $x = 0$, $y = 0$, $y = 3$ Ans: $4\pi \ln 4$

c/ The region enclosed by $x = \sqrt{2 \sin 2y}$, $0 \le y \le \frac{\pi}{2}$, $x = 0$

Ans. 2π

4 Find the volumes of the solids generated by revolving the regions bounded by the lines and curves.

a $y = x$, $y = 1$, $x = 0$ Ans. $\frac{2\pi}{3}$

b $y = x^2 + 1$, $y = x + 3$ Ans: $\frac{117\pi}{5}$

c $y = \sec x$, $y = \sqrt{2}$, $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ Ans: $\pi(\pi - 2)$

5 Find the volume of the solid generated by revolving the following about y-axis

a The region enclosed by the triangle with the vertices $(1,0)$, $(2,1)$ and $(1,1)$ Ans: $\frac{4\pi}{3}$

b The region bounded above by the parabola $y = x^2$ bellow by the x-axis, and on the right by the line $x = 2$. Ans: 8π

6: Find the volume of the solid generated, when the region between the curves of $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

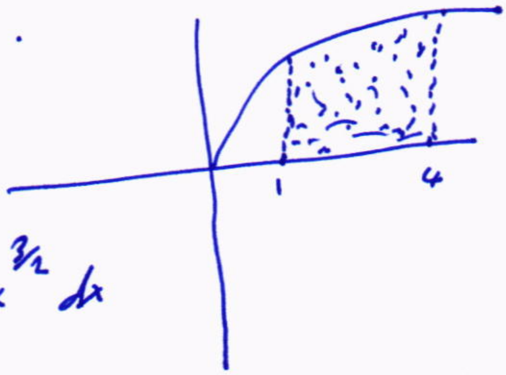
Ans. $\frac{69\pi}{10}$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د. نزار عبد الجليل

3. Volume by Cylindrical shells:

$$V = \int_a^b 2\pi x \, f(x) \, dx.$$

Ex: Use the cylindrical shells to find the volume of the Solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$ and the x-axis is revolved about the y-axis.

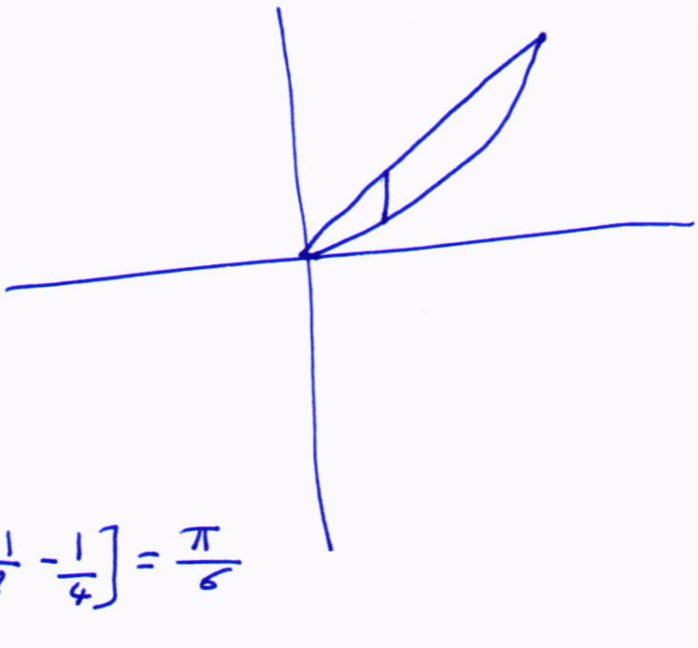


Sol $f(x) = \sqrt{x}$, $a = 1$, $b = 4$

$$V = \int_1^4 2\pi x \sqrt{x} \, dx = 2\pi \int_1^4 x^{3/2} \, dx$$

$$= \left[2\pi \cdot \frac{2}{5} x^{5/2} \right]_1^4 = \frac{4\pi}{5} [32 - 1] = \frac{124\pi}{5}$$

Ex: use cylindrical shells to find the volume of the Solid generated, when the region enclosed by $y = x$, and $y = x^2$ is revolved about the y-axis.



Sol

$$V = \int_0^1 2\pi x (x - x^2) \, dx$$

$$= 2\pi \int_0^1 (x^2 - x^3) \, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{\pi}{6}$$

Ex. Use cylindrical shell to find the volume of the solid generated when the region under $y=x^2$ over the interval $[0,2]$ is revolved about x-axis

Sol

x	y
0	0
2	4

$y=x^2 \Longrightarrow x=\sqrt{y}$

$$V=\int_a^b 2\pi y\left[f(y)-g(y)\right]dx$$

$$V=\int_0^4 2\pi y\left[2-\sqrt{y}\right]dy.$$

$$V=2\pi\int_0^4\left[2y-y^{3/2}\right]dy=2\pi\left[y^2-\frac{y^{5/2}}{5/2}\right]_0^4=\frac{32\pi}{5}$$

H.W.: 1) Use cylindrical shells to find the volumes of the solid generated when the region enclosed by the curves is revolved about y-axis.

a $y=x^3$, $x=1$, $y=0$ Ans: $\frac{2\pi}{5}$

b $x^2+y^3=4$, $x=0$, $x=4$, $y=0$. Ans: $3\pi\sqrt[3]{4}(1+3\sqrt[3]{3})$

c $y=2x-1$, $y=-2x+3$, $x=2$ Ans: $\frac{20\pi}{3}$

2) use cylindrical shells to find the volume of the solid generated when the region enclosed the curve $y^2=x$, $y=1$, $x=0$ is revolved about the x-axis. Ans: $\frac{\pi}{2}$

3 Use the shell method to find the volume of the solids generated by revolving the regions bounded by the curve $x=\sqrt{y}$, $x=-y$, $y=2$ about x-axis Ans: $\frac{16\pi}{15}(3\sqrt{2}+5)$

Arc Length

Length of a curve:

1. if $y = f(x)$, and $\frac{dy}{dx}$ is defined on $[a, b]$, then the arc length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2. if $x = g(y)$, and $\frac{dx}{dy}$ is define on $[c, d]$ then the arc length is:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Ex: Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \le x \le 1$$

Sol $a = 0$, $b = 1$

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2}\, x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left[2\sqrt{2}\, x^{\frac{1}{2}}\right]^2 = 8x.$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big]_0^1 = \frac{13}{6}.$$

Ex Find the length of the graph $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \le x \le 4$

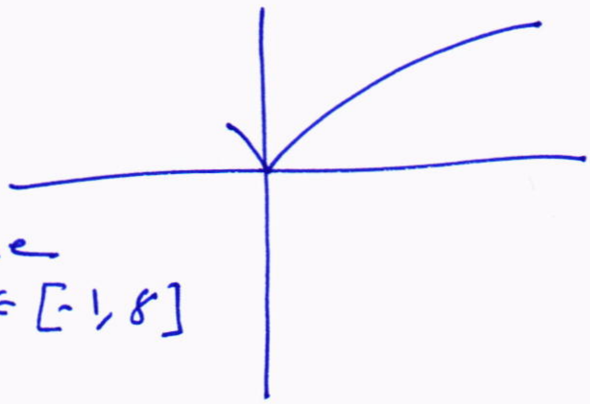
Sol $y' = \frac{x^2}{4} - \frac{1}{x^2}$

So : $L = \int_1^4 \sqrt{1 + (y')^2}\, dx$

$= \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2} \right) dx$

$= \left[\frac{x^3}{12} - \frac{1}{x} \right]_1^4 = \left(\frac{64}{12} - \frac{1}{4} \right) - \left(\frac{1}{12} - 1 \right) = 6$

Ex: Find the arc length of the curve $y = x^{\frac{2}{3}}$ between $x = -1$, and $x = 8$



Sol $y' = \frac{2}{3} x^{-\frac{1}{3}}$

$= \frac{2}{3 x^{\frac{1}{3}}}$ is undefine at $x = 0 \in [-1, 8]$

$y = x^{2/3} \Longrightarrow x = \mp y^{3/2}$

$L_1 : x = -y^{3/2} \Rightarrow y' = -\frac{3}{2} y^{\frac{1}{2}}$, $0 \le y \le 1$

$L_1 = \int_0^1 \sqrt{1 + \frac{9}{4} y}\, dy = \frac{4}{9} \frac{(1 + \frac{9}{4}y)^{3/2}}{3/2} \Big]_0^1 = 1.74$

$L_2 : x = y^{3/2} \Rightarrow \frac{dx}{dy} = \frac{3}{2} y^{\frac{1}{2}}$, $0 \le y \le 4$

$L_2 = \int_0^4 \sqrt{1 + \frac{9}{4} y}\, dy = \frac{4}{9} \frac{(1 + \frac{9}{4}y)^{3/2}}{3/2} \Big|_0^4$

$L = L_1 + L_2$.

H.W

1 Find the length of the curves

a $y = \left(\frac{1}{3}\right)\left(x^2+2\right)^{3/2}$ from $x=0$ to $x=3$: ans 12

b $y = \left(\frac{3}{4}\right)x^{4/3} - \left(\frac{3}{8}\right)x^{2/3} + 5, \quad 1 \le x \le 8$ Ans: $\frac{99}{8}$

2 Find the arc Length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2, 2\sqrt{2})$. Ans $\frac{22\sqrt{22} - 13\sqrt{13}}{27}$

3 Find the arc length of the curve $y = 2x$ from $(1,2)$ to $(2,4)$ Ans $\frac{1}{243}\left(85\sqrt{85} - 8\right)$

4. Find the arc length of the curve :

$$24xy = y^4 + 48 \text{ from } y=2 \text{ to } y=4$$

Ans: $\frac{17}{6}$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الاولى
مساعد أ.د نزاد عبدالحليم

Inverse Function:

if $f: A \longrightarrow B$ then $f^{-1}: B \longrightarrow A$.

or if $y = f(x)$, then $x = f^{-1}(y)$

$D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$

Ex: Let $f(x) = \frac{x}{x-2}$, Find $f^{-1}(x)$

Sol $y = \frac{x}{x-2} \Rightarrow x = yx - 2y \Rightarrow x - yx = -2y$

$x(1-y) = -2y \Rightarrow x = \frac{-2y}{1-y}$

Put x by y we obtain. $\Rightarrow y = \frac{-2x}{1-x}$

$y = \frac{-2x}{1-x} = f^{-1}(x)$.

Note: if $f(x)$ is increase or decrease always. Then f has an inverse.

Ex, Show that $f(x) = x^5 + 7x^3 + 4x + 1$ has an inverse.

$f'(x) = 5x^4 + 21x^2 + 4 > 0$ always increase thus $f(x)$ has an inverse.

Note: $f(x) \circ f^{-1}(x) = f^{-1}(x) \circ f(x) = x$.

Ex: Determine whether $f(x) = \sqrt[3]{x-2}$ is the increase of $g(x) = x^3 + 2$

Sol

$$
\begin{aligned}
f \circ g &= f(g(x)) \\
&= f(x^3+2) \\
&= \sqrt[3]{(x^3+2)-2} \\
&= \sqrt[3]{x^3} \\
&= x
\end{aligned}
\qquad \therefore (f \circ g)_x = x
$$

$\therefore g(x) = f^{-1}(x)$

Logarithms:

Def.: $\log_a x = y$

$\Leftrightarrow \quad x = a^y$

Properties:

1. $\log_a (x \cdot y) = \log_a x + \log_a y$
2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a a = 1, \quad \log_a 1 = 0$

Ex.: $\log_{10} 100 = 2$, since $10^2 = 100$

$\log_2 8 = 3$, since $2^3 = 8$

$\log \frac{1}{1000} = -3$, since $10^{-3} = \frac{1}{1000}$

Theorem:

$\log_b e = \log_e e = 1$ put $b = e$, $e = 2.71828\ldots$

Ex.: Solve for x:

$$\log_5 (5^{2x}) = 8$$

Sol

$2x \log_5 5 = 8$

$2x \cdot 1 = 8$

$\therefore x = 4$

Ex.: Solve $\log_{10} x^2 + \log_{10} x = 30$

Sol

$2 \log_{10} x + \log_{10} x = 30$

$3 \log_{10} x = 30 \Longrightarrow \log_{10} x = 10 \Rightarrow x = 10^{10}$

Note: if $y = \log_a u$ then:

$$\frac{dy}{dx} = \frac{1}{u} \log_a e \cdot \frac{du}{dx} \quad \text{where } e \cong 2.7$$

Ex Find y', if $\log_2 x$

$$\frac{dy}{dx} = \frac{1}{x} \log_2 e.$$

Ex: Find y', if $y = \log_3 (2x^2+5x+1)$

$$\frac{dy}{dx} = \frac{1}{2x^2+5x+1} \log_3 e \cdot (4x+5)$$

$$= \frac{4x+5}{2x^2+5x+1} \log_3 e.$$

Note: $b^{\log_b x} = x$, for $x > 0$.

Natural Logarithm:

if $a = e$ Then $\log_a x = \log_e x = \ln x$

$$y = f(x) = \ln x.$$

$$D_f = \{x : x > 0\}.$$

$$R_f = \{R\}.$$

Ex: Find the domain of $f(x) = \log_3 (x-2)$

Sol

$$x - 2 > 0$$

$$x > 2$$

$$\therefore D_f = \{x : x > 2\}.$$

محاضرات في الرياضيات
مدرس المادة أ.د. نزار عبدالجليل
كلية الهندسة المرحلة الأولى :

Theorem: For any positive numbers a and c and any rational number r.

a. $\ln 1 = 0$

b. $\ln ac = \ln a + \ln c$

c. $\ln \frac{a}{c} = \ln a - \ln c$

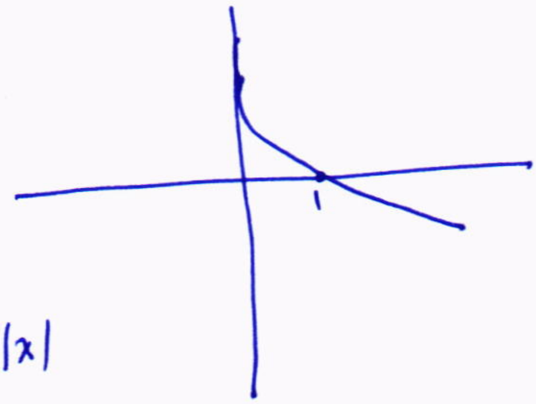
d. $\ln a^r = r \ln a$

e. $\ln \frac{1}{c} = -\ln c$

Ex: Draw the graph of $y = \ln\left(\frac{1}{x}\right)$

Sol

$$\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x$$
$$= 0 - \ln x$$
$$= -\ln x$$

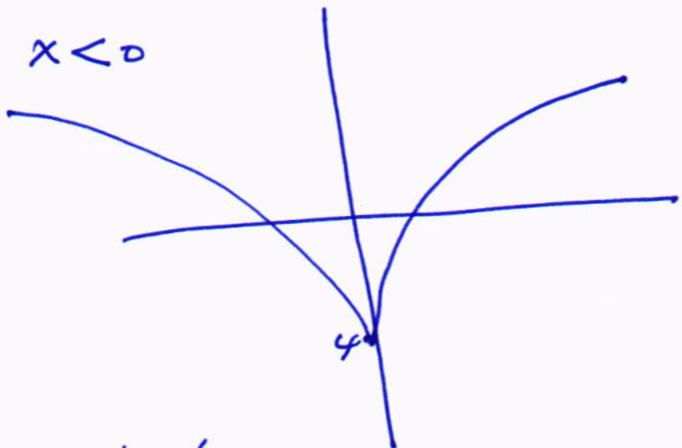


$$\therefore \ln\left(\frac{1}{x}\right) = -\ln x$$

Ex: Draw the graph of $y = \ln|x|$

Sol

$$y = \ln|x| = \begin{cases} \ln x, & x \geq 0 \\ \ln(-x), & x < 0 \end{cases}$$



if $y = \ln u$ then :

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

Ex: Let $y = \ln(\sin x)$ Find y'

Sol

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

Ex: Find $\frac{dy}{dx}$, if $y = \ln\left(\frac{x^2 \sin^3 x}{\cos x^4 \sqrt{1+x}}\right)$

Sol

$$y = \ln\left(\frac{x^2 \sin^3 x}{\cos x^4 \sqrt{1+x}}\right)$$

$$y = \ln(x^2 \sin^3 x) - \ln(\cos x^4 \sqrt{1+x})$$

$$y = \ln x^2 + \ln \sin^3 x - \left[\ln \cos x^4 + \ln(1+x)^{1/2}\right]$$

$$y = 2\ln x + 3\ln \sin x - \ln \cos x^4 - \frac{1}{2}\ln(1+x)$$

$$y' = 2\cdot\frac{1}{x} + 3\cdot\frac{1}{\sin x}\cos x - \frac{1}{\cos x^4}(-\sin x^4\cdot(4x^3)) - \frac{1}{2}\cdot\frac{1}{1+x}$$

$$y' = \frac{2}{x} + 3\cot x + 4x^3 \cot x^2 - \frac{1}{2(1+x)}$$

Ex: Let $y = x^{\sin x}$, Find y'.

Sol

$$y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x}.$$

$$\ln y = \sin x \ln x$$

$$\frac{1}{y}\cdot y' = \sin x \frac{1}{x} + \ln x \cdot \cos x.$$

$$\therefore y' = y\left[\frac{\sin x}{x} + \ln x \cos x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cos x\right]$$

Ex: Find y', if $y = \frac{x^3\sqrt{x-5}}{\sin^2 x \cos^3 x}$.

Sol:

$$\ln y = \ln\left(\frac{x^3\sqrt{x-5}}{\sin^2 x \cos^3 x}\right)$$

$$\ln y = \ln\left[x^3\sqrt{x-5}\right] - \left[\ln \sin^2 x + \ln \cos^3 x\right]$$

$$\ln y = 3\ln x + \frac{1}{2}\ln(x-5) - 2\ln \sin x - 3\ln \cos x.$$

$$\frac{1}{y}\cdot y' = 3\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x-5} - 2\frac{1}{\sin x}\cos x - 3\frac{1}{\cos x}(-\sin x)$$

$$y' = y\left[\frac{3}{x} + \frac{1}{2(x-5)} - 2\cot x + 3\tan x.\right.$$

$$\therefore y' = \frac{x^3\sqrt{x-5}}{\sin^2 x \cos^3 x}\left[\frac{3}{x} + \frac{1}{2(x-5)} - 2\cot x + 3\tan x\right]$$

$$\int \frac{du}{u} = \ln|u| + c.$$

Ex: $\displaystyle\int \frac{x^2\,dx}{x^3-4} = \frac{1}{3}\int \frac{3x^2\,dx}{x^3-4} = \frac{1}{3}\ln|x^3-4| + c.$

Ex. Find $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$

Sol. let $u = 1+\sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{dx}{\sqrt{x}}$$

$$= \int \frac{2\,du}{u} = 2\int \frac{du}{u} = 2\ln|u| + c$$

$$= 2\ln|1+\sqrt{x}| + c.$$

Ex: Find $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx.$

Sol let $u = \ln x \Rightarrow du = \frac{1}{x}.dx$

$$= \int \cos u\,du = \sin u + c$$

$$= \sin(\ln x) + c$$

Ex: $\displaystyle\int \frac{\cot x\,dx}{1} = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$

Ex: $\displaystyle\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$

$$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + c$$

Ex: $\int \frac{\ln x}{x}\,dx$, let $u = \ln x$

$\therefore du = \frac{1}{x}\,dx$.

$= \int u\,du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

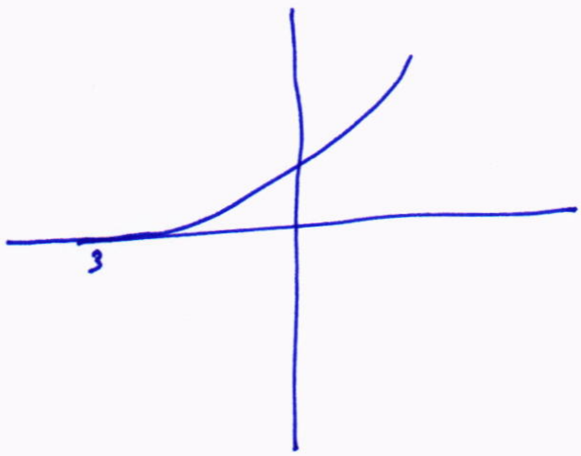
Ex: $\int \frac{dx}{x \ln x}$

Sol let $u = \ln x \Rightarrow du = \frac{1}{x}\cdot dx$

$= \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c$.

Exponential Function:

$y = f(x) = e^x$



$D_f = R$

$R_f = \{y,\ y > 0\}$.

$\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$.

properties:

1. $e^x \cdot e^y = e^{x+y}$
2. $\frac{e^x}{e^y} = e^{x-y}$
3. $(e^x)^n = e^{nx}$.

Note $\boxed{\ln e = 1}$

if $\boxed{y = e^{u} \quad \text{then} \quad y' = e^{u} \cdot \frac{du}{dx}}$

Ex: Find y', if $y = e^{x\tan x}$.

Sol $y' = e^{x\tan x} \cdot [x\sec^2 x + \tan x \cdot 1]$

$$\boxed{\int e^{u}\, du = e^{u} + c}$$

Ex: $\int \frac{e^{\tan x}}{\cos^2 x}\, dx$

Sol $\int e^{\tan x} \sec^2 x\, dx$

let $u = \tan x$, $du = \sec^2 x\, dx$

$= \int e^{u}\, du = e^{u} + c = e^{\tan x} + c$

Ex: $\int e^{2x}\, dx = \int e^{\ln x^2}\, dx = \int x^2\, dx = \frac{x^3}{3} + c.$

Ex: $\int e^{x+e^{x}}\, dx = \int e^{x} \cdot e^{e^{x}}\, dx$, let $u = e^{x} \Rightarrow du = e^{x} \cdot dx$

$= \int e^{u}\, du = e^{u} + c$

$= e^{e^{x}} + c$

$$\boxed{\text{if} \quad y = f(x) = a^{x}, \quad a > 0}$$

1. $a^{x} \cdot a^{y} = a^{x+y}$
2. $\frac{a^{x}}{a^{y}} = a^{x-y}$
3. $(a^{x})^{n} = a^{nx}$

$$\boxed{\text{if } y = a^u, \text{ then } y' = a^u \cdot \frac{du}{dx} \cdot \ln a}$$

Ex: Find y', if $y = 2^{\sin x}$

Sol $\frac{dy}{dx} = 2^{\sin x} \cdot \cos x \cdot \ln 2.$

Ex: Find y', if $y = \pi^{x^2} + e^x \cdot x^e.$

Sol $y' = \pi^{x^2} \cdot (2x) \cdot \ln \pi + [e^x \cdot e\, x^{e-1} + x^e \cdot e^x]$

$$\boxed{\int a^u \, du = \frac{1}{\ln a} \cdot a^u + c}$$

Ex: $\int 2^{\sin x} \cos x \, dx$, Let $u = \sin x \Rightarrow du = \cos x \, dx$

$$= \int 2^u \, du = \frac{1}{\ln 2} \cdot 2^u + c$$

$$= \frac{1}{\ln 2} \cdot 2^{\sin x} + c.$$

Ex: Find $\lim_{x \to \infty} \frac{2 + e^x}{1 + 3e^x}$, $\boxed{\text{note } e^{\infty} = \infty, \; e^{-\infty} = 0}$

Sol $\lim_{x \to \infty} \frac{2 + e^x}{1 + 3e^x} \cdot \frac{e^{-x}}{e^{-x}}$

$$= \lim_{x \to \infty} \frac{2e^{-x} + 1}{e^{-x} + 3} = \frac{0 + 1}{0 + 3} = \frac{1}{3}$$

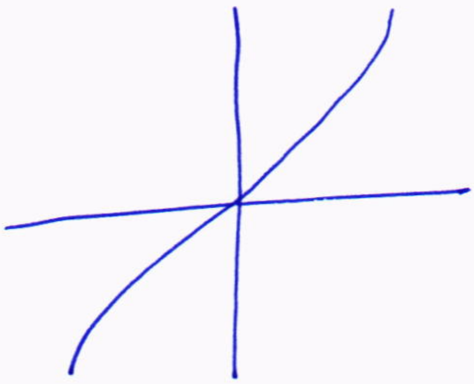
The hyperbolic Functions

Definition:

The hyperbolic sine and hyperpolic cosine functions. denoted by sinh and cosh, respectively, are defined by

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

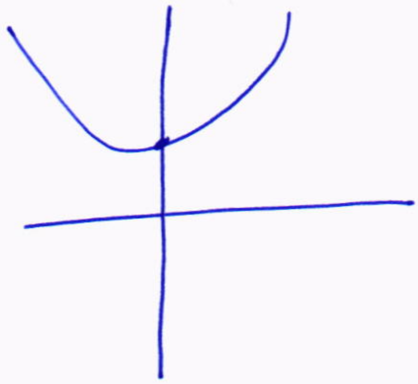
$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$



$y = \sinh x$

$f(x) = \sinh x$

$D_f = R_f = R$



$y = \cosh x$

$f(x) = \cosh x$

$D_f = R$

$R_f : [1, \infty)$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \operatorname{sech}^2 x$
3. $\coth^2 x - 1 = \operatorname{csch}^2 x$
4. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$
5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$
6. $\cosh x + \sinh x = e^{x}$
7. $\cosh x - \sinh x = e^{-x}$
8. $\sinh 2x = 2 \sinh x \cosh x$
9. $\cosh 2x = \begin{cases} \cosh^2 x + \sinh^2 x \\ 2\sinh^2 x + 1 \\ 2\cosh^2 x - 1 \end{cases}$
10. $\cosh(-x) = \cosh x$
11. $\sinh(-x) = -\sinh x.$

Derivative	Integrals
1. $(\sinh x)' = \cosh x$	1- $\int \cosh x \, dx = \sinh x + c$
2. $(\cosh x)' = \sinh x$	2- $\int \sinh x \, dx = \cosh x + c$
3. $(\tanh x)' = \operatorname{sech}^2 x$	3- $\int \operatorname{sech}^2 x \, dx = \tanh x + c$
4. $(\coth x)' = -\operatorname{csch}^2 x$	4. $\int \operatorname{csch}^2 x \, dx = -\coth x + c$
5. $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$	5. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$
6. $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$	6. $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + c$

Ex: prove that.

$\cosh(3x) = 4\cosh^3 x - 3\cosh x$

Sol $\cosh 3x = \cosh(x+2x)$

$= \cosh x \cosh 2x + \sinh x \sinh 2x$

$= \cosh x\,(2\cosh^2 x - 1) + \sinh x\,(2\sinh x \cosh x)$

$= 2\cosh^3 x - \cosh x + 2\sinh^2 x \cosh x$

$= 2\cosh^3 x - \cosh x + 2(\cosh^2 x - 1)\cosh x$

$= 2\cosh^3 x - \cosh x + 2\cosh^3 x - 2\cosh x$

$= 4\cosh^3 x - 3\cosh x$

Ex: Solve $\sinh x = \frac{1}{4}\cosh x$ in terms of $\ln 3$, $\ln 5$.

Sol $\sinh x = \frac{1}{4}\cosh x$

$$\frac{e^x - e^{-x}}{2} = \frac{1}{4}\cdot\frac{e^x + e^{-x}}{2}$$

$$\left[\frac{e^x - e^{-x}}{2} = \frac{1}{4}\cdot\frac{e^x + e^{-x}}{2}\right] \times 2$$

$$\left[e^x - e^{-x} = \frac{1}{4}e^x + \frac{1}{4}e^{-x}\right] \times e^x$$

$$e^{2x} - 1 = \frac{1}{4}e^{2x} + \frac{1}{4}$$

$$e^{2x} - \frac{1}{4}e^{2x} = 1 + \frac{1}{4}$$

$$\left[\frac{3}{4}e^{2x} = \frac{5}{4}\right] \times \frac{4}{3}$$

$$e^{2x} = \frac{5}{3} \Rightarrow 2x = \ln\frac{5}{3}$$

$$x = \frac{1}{2}[\ln 5 - \ln 3]$$

Ex: Find $\frac{d}{dx}\left[\int_0^{\ln x} \frac{dt}{\sqrt{4+e^t}}\right]$

Sol By The second Fundamental Theorem of Integral: which say that

$$\boxed{\text{if } F(x) = \int_a^x f(t)\,dt \text{ , then } F'(x) = f(x)}$$

$t = \ln x \Rightarrow dt = \frac{1}{x}$, $e^{\ln x} =$

$$= \frac{1}{\sqrt{4+e^{\ln x}}} \cdot \frac{1}{x} = \frac{1}{x\sqrt{4+x}}$$

Ex: Show that $y = e^{ax}\sinh bx$ satisfies

$$y'' - 2ay' + (a^2+b^2)y = 0.$$

Sol $y' = b\,e^{ax}\cosh x\,bx + a\,e^{ax}\sinh bx.$

$$y'' = b^2 e^{ax}\sinh bx + ab\,e^{ax}\cosh bx + ab\,e^{ax}\cosh bx + a^2 e^{ax}\sinh bx$$

$$y'' = e^{ax}\sinh bx\,(a^2+b^2) + 2ab\,e^{ax}\cosh bx$$

$$y'' - 2ay' + (a^2+b^2)y = (a^2+b^2)e^{ax}\sinh bx + 2ab\,e^{ax}\cosh bx$$
$$-2a\left[b\,e^{ax}\cosh x + a\,e^{ax}\sinh bx\right] + (a^2+b^2)\,e^{ax}\sinh bx) = 0.$$

Ex: Evaluate : $\int\left[\tanh x + \sqrt{\tanh x}\cdot\operatorname{sech}^2 x\right]dx$

$$= \int \frac{\sinh x}{\cosh x}\,dx + \int (\tanh x)^{1/2}\operatorname{sech}^2 x\,dx$$

$$= \ln|\cosh x| + \frac{(\tanh x)^{3/2}}{3/2} + c$$

Ex: Evaluate: $\int \frac{\sinh 2x\,dx}{3+5\cosh 2x}$

Let $u = 3+5\cosh 2x \Rightarrow du = 10\sinh 2x\,dx$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
إعداد أ.د. زياد عبد الجليل

$$\int \frac{\sinh 2x\,dx}{3+5\cosh 2x}$$

$$= \frac{1}{10}\int \frac{du}{u} = \frac{1}{10}\ln|u| + c$$

$$= \frac{1}{10}\ln|3+5\cosh 2x| + c.$$

Ex: show that: $\lim\limits_{x\to\infty} \tanh x = 1$

Sol $\lim\limits_{x\to\infty} \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$$= \lim_{x\to\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \cdot \frac{e^{x}}{e^{x}}$$

$$= \lim_{x\to\infty} \frac{e^{2x}-1}{e^{2x}+1} = -1$$

Inverse trigonometric and hyperbolic functions.

if $-1 \le x \le 1$, and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, then $y = \sin^{-1} x$ and $\sin y = x$

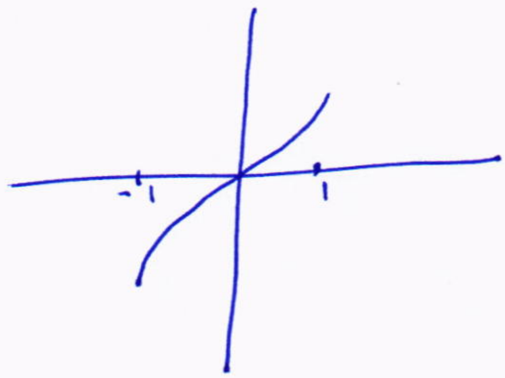
$\sin^{-1}(\sin y) = y$, if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\sin(\sin^{-1} x) = x$, if $-1 \le x \le 1$

1) $f(x) = \sin^{-1} x$

$D_f = [-1, 1]$

$R_f = [-\frac{\pi}{2}, \frac{\pi}{2}]$.



Ex: Simplify:

$\cos(\sin^{-1} x)$

Sol let $\alpha = \sin^{-1} x \Rightarrow \frac{x}{1} = \sin\alpha$

$\therefore \cos(\sin^{-1} x) = \cos\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$

Ex: Simplify:

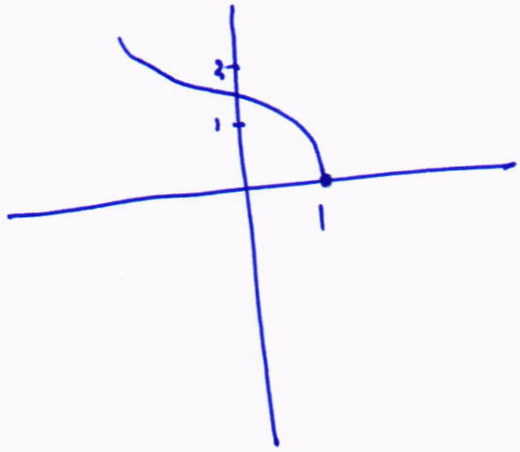
$\sec^2(\tan^{-1} x)$

Sol let $\alpha = \tan^{-1} x \Rightarrow \tan\alpha = \frac{x}{1}$

$\therefore \sec(\tan^{-1} x) = \sec\alpha = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$

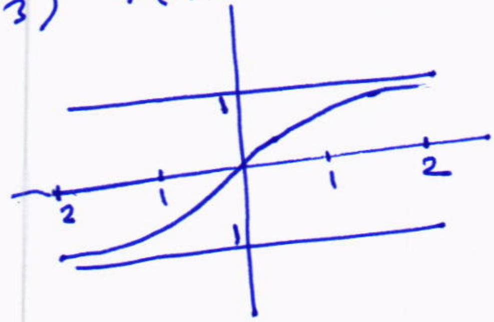
$= \sec^2(\tan^{-1} x) = 1+x^2.$

2) $f(x) = \cos^{-1} x$



$D_f = [-1,1]$, $R_f = [0,\pi]$

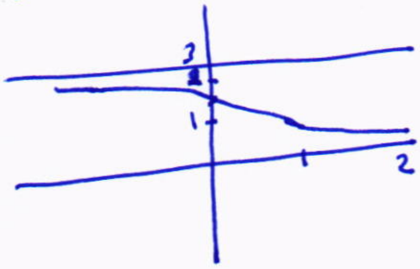
3) $f(x) = \tan^{-1} x$



$D_f = R$

$R_f = (-\frac{\pi}{2}, \frac{\pi}{2})$

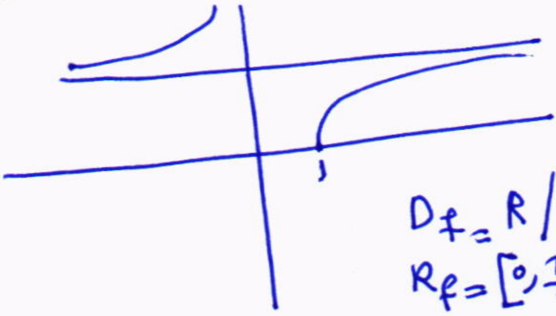
4) $f(x) = \cot^{-1} x$



$D_f = R$

$R_f = (0, \pi).$

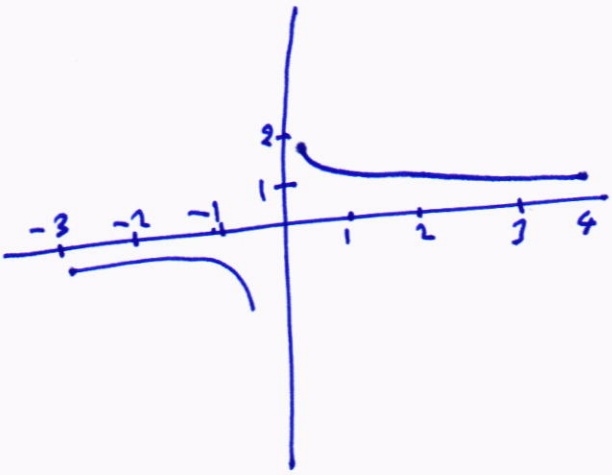
5) $f(x) = \sec^{-1} x$



$D_f = R / (-1,1)$

$R_f = [0,\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

6) $f(x) = \csc^{-1} x$



$D_f = R / (-1,1)$

$R_f = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}].$

Ex: prove that:

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

Sol let $y=\cos^{-1}x \Rightarrow x=\cos y$

$$x=\sin\left(\frac{\pi}{2}-y\right)$$

$$\sin^{-1}x=\frac{\pi}{2}-y.$$

$y=\frac{\pi}{2}-\sin^{-1}x \qquad \therefore \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$

or $\quad \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}.$

Ex: Find the value of $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$.

Sol let $\alpha=\cos^{-1}\frac{3}{5} \Leftrightarrow \cos\alpha=\frac{3}{5}$

$$\therefore \sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)=\sin 2\alpha$$

$$=2\sin\alpha\cos\alpha$$

$$=2\cdot\frac{4}{5}\cdot\frac{3}{5}=\frac{24}{25}$$

Ex: find the value of $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)+\cos^{-1}\left(\frac{1}{3}\right)\right]$

Sol: let $\alpha=\sin^{-1}\frac{2}{3}$, $\beta=\cos^{-1}\left(\frac{1}{3}\right)$.

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$\sin\alpha=\frac{2}{3}$, $\cos\beta=\frac{1}{3}$

$$\sin[\alpha+\beta]=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

$$=\frac{2}{3}\cdot\frac{1}{3}+\frac{\sqrt{5}}{3}\cdot\frac{\sqrt{8}}{3}$$

$$=\frac{2}{9}+\frac{\sqrt{40}}{9}=\frac{2+\sqrt{40}}{9}$$

Ex: prove that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Sol let $\alpha=\tan^{-1}x \Rightarrow \tan\alpha=x$

$\beta=\tan^{-1}y \Rightarrow \tan\beta=y$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$\therefore \alpha + \beta = \tan^{-1}\left(\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right)$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Derivatives and Integrals of inverse trigonometric Functions

Theorem:

1. $(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$
2. $(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$
3. $(\tan^{-1}x)' = \frac{1}{1+x^2}$
4. $(\cot^{-1}x)' = -\frac{1}{1+x^2}$
5. $(\sec^{-1}x)' = \frac{1}{x\sqrt{x^2-1}}$
6. $(\csc^{-1}x)' = \frac{1}{x\sqrt{x^2-1}}$

Integrals

1. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$
2. $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$
3. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c.$

Ex: Find $\frac{dy}{dx}$, if $y = \sin^{-1}(x^3)$.

Sol:

$$y' = \frac{1}{\sqrt{1-(x^3)^2}} \cdot (3x^2)$$

$$= \frac{3x^2}{\sqrt{1-x^6}}$$

Ex: Find $\frac{dy}{dx}$, if $y = \sec^{-1}e^x$.

Sol:

$$y' = \frac{1}{e^x\sqrt{(e^x)^2-1}}\, e^x = \frac{1}{\sqrt{e^{2x}-1}}$$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من إعداد أ.د نزار عبد الجليل

Ex: Evaluate.

$$\int \frac{dx}{1+3x^2}$$

Sol Substituting:

$u=\sqrt{3}\,x$, $du=\sqrt{3}.dx$

$$\Rightarrow \int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}} \int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}} \tan^{-1} u + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\,x) + c.$$

Ex: Evaluate:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$$

Sol: let: $u=e^x$, $du=e^x.\,dx$.

$$\Rightarrow \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = \sin^{-1}(e^x) + c$$

Ex Evaluate: $\int \frac{dx}{a^2+x^2}$, where $a \neq 0$ is a constant

Sol let $x = au \Rightarrow dx = a\,du$.

then $$\int \frac{dx}{a^2+x^2} = \int \frac{a\,du}{a^2+a^2u^2} = \frac{1}{a} \int \frac{du}{1+u^2}$$

$$= \frac{1}{a} \tan^{-1} u + c = \frac{1}{a} \tan^{-1} \frac{x}{a} + c.$$

Ex: Evaluate:

$$\int \frac{dx}{\sqrt{2-x^2}}$$, By Method of above example.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$, $a>0$.

$$\therefore \int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$$

Ex: $\int \frac{dx}{\sqrt{9-x^2}}$

Sol $\int \frac{dx}{\sqrt{9(1-\frac{x^2}{9})}} = \int \frac{dx}{3\sqrt{1-(\frac{x}{3})^2}}$

let $u = \frac{x}{3} \Rightarrow du = \frac{1}{3}dx$

$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = \sin^{-1}(\frac{x}{3}) + c.$

Inverse of Hyperbolic Functions

Ex prove that: $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

Sol let $y = \sinh^{-1} x \Rightarrow \sinh y = x$

$\frac{e^y - e^{-y}}{2} = x$

$[e^y - e^{-y} = 2x] \times e^y$

$e^{2y} - 1 = 2xe^y.$

$1.\ e^{2y} - 2xe^y - 1 = 0$

$a = 1, \quad b = -2x, \quad c = -1$

$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$e^y = \frac{2x + 2\sqrt{x^2+1}}{2}$

$e^y = x \pm \sqrt{x^2+1}$, since $e^y > 0$

$e^y = x + \sqrt{x^2+1}$

$y = \ln(x + \sqrt{x^2+1}$

$\therefore \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

Ex: Find $\frac{dy}{dx}$ of

1) $y = 2^x + \sinh^{-1}(x^2)$

$$\frac{dy}{dx} = 2^x \cdot \ln 2 + \frac{1}{\sqrt{1+x^4}} \cdot 2x$$

2) $y = (1 + x \operatorname{csch}^{-1} x)^{10}$

$$y' = 10(1 + x \operatorname{csch}^{-1} x)^9 \cdot \left[x \cdot \frac{-1}{|x|\sqrt{1+x^2}} + \operatorname{csch}^{-1} x \cdot 1\right]$$

Ex: show that if $y = \tan^{-1} x$ then $y'' = -2 \sin y \cos^3 y$

Sol $y = \tan^{-1} x \Rightarrow \tan y = x$

$$\sec^2 y \, y' = 1$$

or $y' = \frac{1}{\sec^2 y} = \cos^2 y$

or $y' = [\cos y]^2$.

$$y'' = 2[\cos y] - (-\sin y)\, y'$$

$$y'' = 2[\cos y] - (-\sin y)(\cos^2 y)$$

$$\therefore y'' = -2 \sin y \cos^3 y.$$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د نهاد عبد الجليل

Techniques of Integration

Trigonometric Substitutions.

1. if $\sqrt{a^2-u^2}$ then Let $u = a\sin\theta$.
2. if $\sqrt{a^2+u^2}$ then Let $u = a\tan\theta$.
3. if $\sqrt{u^2-a^2}$ then Let $u = a\sec\theta$.

Ex: Evaluate: $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

Sol: let $x = 2\sin\theta. \Rightarrow dx = 2\cos\theta\, d\theta$.

$$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

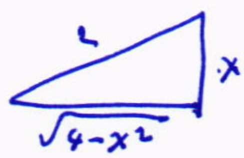
$$= \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\sin^2\theta \cdot 2\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\cos^2\theta}} = \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + c.$$

$\because x = 2\sin\theta \Rightarrow \frac{x}{2} = \sin\theta$.



$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + c.$$

Ex: Evaluate: $\displaystyle\int \frac{dx}{(3+x^2)^{3/2}}$

Sol let $x = \sqrt{3}\tan\theta \Rightarrow dx = \sqrt{3}\sec^2\theta\, d\theta$

$\frac{x}{\sqrt{3}} = \tan\theta$

$$= \int \frac{\sqrt{3}\sec^2\theta\, d\theta}{(3+3\tan^2\theta)^{3/2}}$$

$$= \sqrt{3} \int \frac{\sec^2\theta \, d\theta}{3^{3/2}(1+\tan^2\theta)^{3/2}}$$

$$= \frac{1}{3}\int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^{3/2}} = \frac{1}{3}\int \frac{\sec^2\theta \, d\theta}{\sec^3\theta}.$$

$$= \frac{1}{3}\int \frac{d\theta}{\sec\theta} = \frac{1}{3}\int \cos\theta \, d\theta = \frac{1}{3}\sin\theta + c$$

$$= \frac{1}{3}\frac{x}{\sqrt{9+x^2}} + c.$$

$$\boxed{\text{Integration involving } ax^2+bx+c}$$

Ex: Evaluate:

$$\int \frac{dx}{\sqrt{5-4x-2x^2}}.$$

Sol

$$5-4x-2x^2 = -2(x^2+2x)+5$$

$$= -2(x^2+2x+1-1)+5$$

$$= -2[(x+1)^2-1]+5.$$

$$= -2(x+1)^2+2+5$$

$$= 7-2(x+1)^2$$

$$\int \frac{dx}{\sqrt{5-4x-2x^2}} = \int \frac{dx}{\sqrt{7-2(x+1)^2}}$$

Let $u = x+1 \Rightarrow du = dx$

Let $\sqrt{2}\,u = \sqrt{7}\sin\theta \Longrightarrow du = \frac{\sqrt{7}}{\sqrt{2}}\cos\theta \, d\theta.$

$$= \int \frac{du}{\sqrt{7-2u^2}} = \sqrt{\frac{7}{2}}\int \frac{\cos\theta \, d\theta}{\sqrt{7-7\sin^2\theta}} = \frac{\sqrt{7}}{\sqrt{2}}\int \frac{\cos\theta \, d\theta}{\sqrt{7}\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos\theta \, d\theta}{\sqrt{\cos^2\theta}} = \frac{1}{\sqrt{2}} \int d\theta$$

$$= \frac{1}{\sqrt{2}} \theta + c$$

$$\because \frac{\sqrt{2}}{\sqrt{7}} u = \sin\theta \Rightarrow \sin^{-1}\left(\sqrt{\tfrac{2}{7}}\, u\right) = \theta.$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\tfrac{2}{7}}\, u + c\right)$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\tfrac{2}{7}}\,(x-3)\right) + c.$$

Integration by parts

$$\int u\, dv = uv - \int v\, du.$$

<u>Ex.</u> Evaluate: $\int x^2 . e^{-x}\, dx$

<u>Sol</u> let $u = x^2 \Rightarrow du = 2x\, dx$

$dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = e^{-x}$

So that $\int x^2 e^{-x}\, dx = \int u\, dv = uv - \int v\, du$

$$= -x^2 e^{-x} - 2\int x e^{-x}\, dx$$

to find $\int x e^{-x} dx$

let $u = x \Rightarrow du = dx$

$dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}.$

So that $\int x e^{-x} dx = \int u\, dv = uv - \int v\, du.$

$$= -x e^{-x} + \int e^{-x} dx$$

$$= -x e^{-x} - e^{-x} + c_1$$

$$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} + 2\left(-x e^{-x} - e^{-x} + c_1\right)$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + 2c_1$$

$$= -(x^2 + 2x + 2) e^{-x} + c \quad , \quad c = 2c_1$$

Ex: Evaluate.

$$\int \ln x \, dx$$

Sol: let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = dx \Rightarrow v = \int dx = x$

So that $\int \ln x \, dx = \int u \, dv = uv - \int v \, du$

$= x \ln x - \int x \frac{1}{x} dx$

$= x \ln x - \int dx = x \ln x - x + c$

Ex: Evaluate: $\int e^x \cos x \, dx$.

Sol let $u = e^x \Rightarrow du = e^x dx$

$dv = \cos x \, dx \Rightarrow v = \sin x$.

$\therefore \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

let $u = e^x \Rightarrow du = e^x dx$

$dv = \sin x \, dx \Rightarrow v = -\cos x$

$\int e^x \cos x \, dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x \, dx]$

$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$.

$\therefore \int e^x \cos x \, dx = \dfrac{e^x \sin x + e^x \cos x}{2} + c$

Ex: Evaluate $\int x^3 e^x \, dx$

Sol By using tabular method:

$+x^3$	e^x
$-3x^2$	e^x
$+6x$	e^x
-6	e^x
0	e^x

$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c$

Ex: Evaluate $\int \frac{x e^x dx}{(x+1)^2}$

Sol let $u = xe^x \Rightarrow du = (xe^x + e^x)\,dx$

$dv = (x+1)^{-2}\,dx \Rightarrow v = \frac{-1}{x+1}$

$du = e^x(x+1)\,dx$

$$\int \frac{x e^x dx}{(x+1)^2} = \frac{-xe^x}{(x+1)} + \int \frac{e^x(x+1)\,dx}{x+1}$$

$$= \frac{-xe^x}{(x+1)} + \int e^x\,dx$$

$$= \frac{-xe^x}{(x+1)} + e^x + c$$

Integration by partial Fraction

Ex: Evaluate $\int \frac{x^3\,dx}{x^2-3x+2}$

Sol

$$\begin{array}{r|l} & x+3 \\ x^2-3x+2 & x^3 \\ & \mp x^3 \pm 3x^2 \mp 2x \\ \hline & 3x^2-2x \\ & 3x^2 \pm 9x \mp 6 \\ \hline & 7x-6 \end{array}$$

$$= \int \left[x+3 + \frac{7x-6}{x^2-3x+2}\right] dx$$

$$= \frac{x^2}{2} + 3x + \int \frac{7x-6}{(x-1)(x-2)}\,dx$$

$$\frac{7x-6}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$\frac{7x-6}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$7x-6 = Ax-2A+Bx-B$$

$$7x-6=(A+B)x-2A-B$$

$$\therefore A+B=7$$

$$-2A-B=-6$$

$$\therefore A=-1 \quad , \quad B=8$$

$$\int \frac{(7x-6)\,dx}{(x-1)(x-2)} = \int \frac{-1\,dx}{x-1} + \int \frac{8\,dx}{x-2}$$

$$= -\ln|x-1| + 8\ln|x-2| + c.$$

$$\therefore \int \frac{x^3\,dx}{x^2-3x+2} = \frac{x^2}{2} + 3x - \ln|x-1| + 8\ln|x-2| + c$$

<u>Ex:</u> Evaluate $\int \frac{\cos\theta\, d\theta}{\sin^2\theta + 4\sin\theta - 5}$

<u>Sol</u> Let $x = \sin\theta \Rightarrow dx = \cos\theta \cdot d\theta.$

$$= \int \frac{dx}{x^2-4x-5} = \int \frac{dx}{(x+1)(x-5)}$$

$$\frac{1}{(x+1)(x-5)} = \frac{A}{(x+1)} + \frac{B}{(x-5)}$$

$$\therefore A = \frac{-1}{6} \quad , \quad B = \frac{1}{6}$$

$$\int \frac{dx}{(x+1)(x-5)} = -\frac{1}{6}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{dx}{x-5}$$

$$= \frac{-1}{6}\ln|x+1| + \frac{1}{6}\ln|x-5| + c$$

$$= -\frac{1}{6}\ln|\sin\theta + 1| + \frac{1}{6}\ln|\sin\theta| + c$$

Def.: Any equation involve derivative is called Differential equation

Ex:
1) $\frac{dy}{dx} = \frac{x}{y}$
2) $(x+y)\,dy = xy\,dx$

are D. Es.

Ex.: **Solve:** D.E. $\frac{dy}{dx} = y^2 - 5y + 6$

Sol

$$\int \frac{dy}{y^2 - 5y + 6.} = \int dx$$

$$= \int \frac{dy}{(y-2)(y-3)} = \int dx$$

$$\frac{1}{(y-2)(y-3)} = \frac{A}{(y-2)} + \frac{B}{(y-3)}$$

$$\therefore A = -1 \quad , \quad B = 1$$

$$\int \frac{A\,dy}{(y-2)} + \int \frac{B\,dy}{(y-3)} = \int dx$$

$$= \frac{\int -1\,dy}{(y-2)} + \int \frac{dy}{y-3} = \int dx$$

$$-\ln|y-2| + \ln|y-3| = x \Rightarrow \ln\left|\frac{y-3}{y-2}\right| = x$$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د. نزار عبد الجليل

Integrals involving rational exponents

<u>Ex:</u> Evaluate $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

<u>Sol</u> let $u = x^{\frac{1}{6}} \Rightarrow u^6 = x \Rightarrow dx = 6u^5\,du.$

$$\int \frac{6u^5\,du}{u^3+u^2}$$

$$= 6\int \frac{u^5\,du}{u^2(u+1)} = 6\int \frac{u^3\,du}{u+1}$$

$$\begin{array}{r|l} & u^2-u+1 \\ \hline u+1 & u^3 \\ & \mp u^3 \mp u^2 \\ \hline & -u^2 \\ & \mp u^2 \mp u \\ \hline & +u \\ & \mp u \mp 1 \\ \hline & -1 \end{array}$$

$$= 6\int \left[u^2 - u + 1 - \frac{1}{u+1}\right] du$$

$$= 6\int \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + c$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln\left|x^{\frac{1}{6}}+1\right| + c.$$

<u>Ex</u> $\displaystyle\int \frac{dx}{x - x^{3/5}}$

let $u = x^{\frac{1}{5}} \Rightarrow u^5 = x \Rightarrow 5u^4\,du = dx$

$$\int \frac{5u^4\,du}{u^5-u^3} = 5\int \frac{u^4\,du}{u^3(u^2-1)} = 5\int \frac{u\,du}{u^2-1}$$

$$= \frac{5}{2}\int \frac{2u\,du}{(u^2-1)} = \frac{5}{2}\ln\left|u^2-1\right| + c$$

$$= \frac{5}{2}\ln\left|x^{2/5}-1\right| + c$$

Integrals involving rational expression in sin and cos

Ex Evaluate $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

Sol Let $x = 2\tan^{-1} u \Rightarrow dx = \dfrac{2du}{1+u^2}$

$\sin x = \dfrac{2u}{1+u^2}$, $\cos x = \dfrac{1-u^2}{1+u^2}$

$$\int \frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} = \int \frac{\frac{2du}{1+u^2}}{\frac{1+u^2+2u+1-u^2}{1+u^2}}$$

$$= \int \frac{2du}{2+2u} = \int \frac{du}{1+u}$$

$$= \ln|1+u| + c = \ln\left|1+\tan\frac{x}{2}\right| + c$$

Integration power of sine and cosine

Ex: Evaluate: $\displaystyle\int \sin^3 x \cos^2 x \, dx$

The power is odd and even.

$$= \int \sin^2 x \sin x \cos^2 x \, dx$$

$$= \int (1-\cos^2 x)\sin x \cos^2 x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$= -\int (1-u^2)u^2 \, du = -\int (u^2 - u^4)\, du$$

$$= \frac{-u^3}{3} + \frac{u^5}{5} + c = \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$= \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$= \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

__Ex:__ Evaluate $\int \sin 7x \cos 3x \, dx$

__Sol__ $\int \frac{1}{2}\left[\sin(4x) + \sin(10x)\right] dx$

$$= \frac{1}{2}\int \left[\sin(4x)\right] dx + \frac{1}{2}\int \left[\sin(10x)\right] dx$$

$$= \frac{1}{8}\int \left[\sin(4x)\right] 4 \cdot dx + \frac{1}{20}\int \sin(10x) \cdot 10 \, dx$$

$$= \frac{-1}{8}\cos 4x - \frac{1}{20}\cos 10x + c.$$

__Ex:__ Evaluate $\int \sin x \cos \frac{x}{2} \, dx$

$$= \int \frac{1}{2}\left[\sin\left(\frac{x}{2}\right) + \sin\left(\frac{3x}{2}\right)\right] dx$$

$$= \frac{1}{2}\int \left[\sin\left(\frac{x}{2}\right)\right] dx + \frac{1}{2}\int \left[\sin\left(\frac{3x}{2}\right)\right] dx$$

$$= -\cos\frac{x}{2} - \frac{1}{3}\cos\frac{3x}{2} + c.$$

$\boxed{\text{Integrating power of secant and tangent}}$

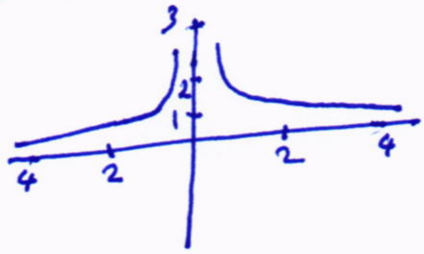
Reduction formula.

$$\int \sec^n x \, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

$$\int \tan^n x \, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \, dx$$

Improper Integrals

1) $\int_{a}^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$ are called improper integrals.



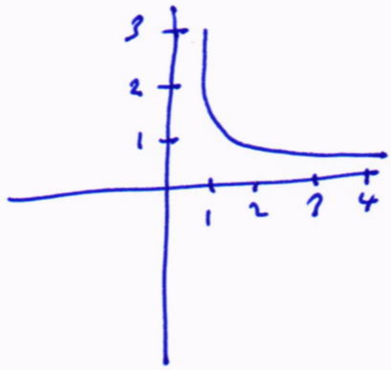
Ex: Find $\int_{1}^{\infty} \frac{dx}{x^2}$

Sol $\lim\limits_{b \to \infty} \int_{1}^{b} \frac{dx}{x^2} = \lim\limits_{b \to \infty} \left[-\frac{1}{x} \right]_{1}^{b}$

$= \lim\limits_{b \to \infty} \left[-\frac{1}{b^2} + 1 \right] = \lim\limits_{b \to \infty} \left[1 - \frac{1}{b} \right] = 1$

and the integral is converges to 1.

Ex: Evaluate $\int_{1}^{\infty} \frac{dx}{x}$



Sol $\lim\limits_{b \to \infty} \int_{1}^{b} \frac{dx}{x}$

$= \lim\limits_{b \to \infty} \left[\ln x \right]_{1}^{b}$

$= \lim\limits_{b \to \infty} \left[\ln b - \ln 1 \right] = \infty$

The integral is diverges

Ex: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Sol $\int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$= \lim\limits_{b \to -\infty} \int_{b}^{0} \frac{dx}{1+x^2} + \lim\limits_{a \to \infty} \int_{0}^{a} \frac{dx}{1+x^2}$

$$= \lim_{b \to -\infty} \left[\tan^{-1} x \right]_b^0 + \lim_{a \to \infty} \tan^{-1} x \Big]_0^a$$

$$= \lim_{b \to -\infty} \left[\tan^{-1} 0 + \tan^{-1} b \right] + \lim_{a \to \infty} \left[\tan^{-1} a - \tan^{-1} 0 \right]$$

$$= -\left(\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

the integrals is converges to π.

Ex: Evaluate: $\int_1^4 \frac{dx}{(x-2)^{2/3}}$, this integral is improper because: $f(x) \to \infty$ as $x \to 2$ and $2 \in [1, 4]$.

Sol $\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$

$$= \lim_{a \to -2} \int_1^a (x-2)^{-2/3} dx + \lim_{b \to +2} (x-2)^{-2/3} dx \Big]_b^4$$

$$= \lim_{a \to -2} \left[3(x-2)^{1/3} \right]_1^a + \lim_{b \to +2} \left[3(x-2)^{1/3} \right]_b^4$$

$$= \lim_{a \to -2} \left[3(a-2)^{1/3} - 3\right](-1)^{1/3} \Big] + \lim_{b \to 2} \left[3(2)^{1/3} - 3[b-2]^{1/3} \right]$$

$$= 3\sqrt[3]{2}$$

the integral is converges.

L'HÔPITAL'S RULE

Theorem:

if $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Ex: Find $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$ $\left(\frac{0}{0}\right)$

Sol

by L'HÔPITAL Rule

$$\lim_{x \to 0} \frac{2\cos 2x}{1} = 2 \cdot 1 = 2$$

Ex: Find $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x^3}$ $\left(\frac{0}{0}\right)$

by L'hopital Rule:

$$\lim_{x \to 0} \frac{e^x}{3x^2} = \frac{1}{0} = \infty$$

The $\lim$ does not exist.

Ex: $\lim\limits_{x \to \infty} \dfrac{x^2 + x - 1}{2x^2 + 5}$ $\left(\frac{\infty}{\infty}\right)$

Sol by l'hopital Rule.

$$= \lim_{x \to \infty} \frac{2x+1}{4x} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{2}{4} = \frac{1}{2}.$$

Ex: Find $\lim\limits_{x \to \infty} \dfrac{\ln\left(1 + \frac{3}{x}\right)}{\sin\left(\frac{2}{x}\right)}$ $\left(\frac{0}{0}\right)$

Sol by hopital Rule.

$$\lim_{x \to \infty} \frac{\frac{1}{1+\frac{3}{x}} \cdot \frac{-3}{x^2}}{\cos\left(\frac{2}{x}\right) \cdot \frac{-2}{x^2}} = \frac{3}{2}$$

$$\boxed{\text{The Form } (0,\infty) \rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$$

Ex: Find $\lim_{x \to 0^+} x \ln x \quad (0, \infty)$.

Sol

$$= \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$$

by HOPITAL Rule.

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} (-x) = 0.$$

$$\boxed{\text{The Form } (+\infty, -\infty) \rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$$

Ex: Find $\lim_{x \to \infty} \left(\sqrt{x^2+x} - x\right) \quad (+\infty, -\infty)$

Sol

$$\lim_{x \to \infty} \left(\sqrt{x^2+x} - x\right) \cdot \frac{\sqrt{x^2+x} + x}{\sqrt{x^2+x} + x}$$

$$= \lim_{x \to \infty} \frac{x^2 + x - x^2}{\sqrt{x^2+x} + x} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2+x} + x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \frac{1}{2}$$

Ex: Find $\lim_{x \to \infty} \left[x - \ln(x^2+1)\right] \quad (\infty, -\infty)$

Sol

$$\lim_{x \to \infty} \left[\ln e^x - \ln(x^2+1)\right]$$

$$= \lim_{x \to \infty} \left[\ln \frac{e^x}{x^2+1}\right]$$

$$= \ln\left[\lim_{x \to \infty} \frac{e^x}{x^2+1}\right] \quad \frac{\infty}{\infty} \quad \text{by HÔPITAL Rule}$$

$$= \ln\left[\lim_{x \to \infty} \frac{e^x}{2x}\right] \quad \frac{\infty}{\infty} \quad \text{by HÔPITAL Rule}$$

$$= \ln\left[\lim_{x \to \infty} \frac{e^x}{2}\right] = \ln \infty = \infty$$

The Form $\left(\frac{0}{0}, \frac{\infty}{\infty}, 1^{\infty}\right)$

Ex: Find $\lim_{x \to 0^+} (1+x)^{1/x}$ $\quad 1^{\infty}$

Sol: let $y = (1+x)^{\frac{1}{x}}$.

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{1}{x} \ln(1+x) \quad (0, \infty)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \quad \frac{0}{0} \text{ by HÔPITAL Rule}$$

$$\ln \lim_{x\to 0^+} y = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

$$\ln \lim_{x\to 0^+} y = 1$$

take e for both sides

$$\lim_{x\to 0^+} y = e^1 \longrightarrow \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e.$$

Ex Find $\lim_{x\to 0^+} (e^x + x)^{1/x}$

Sol let $y = (e^x + x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(e^x + x)$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{1}{x} \ln(e^x + x) \quad (0, \infty)$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(e^x + x)}{x} \quad \left(\frac{0}{0}\right) \text{ by Hopital Rule}$$

$$\ln \lim_{x\to 0^+} y = \lim_{x\to 0^+} \frac{\frac{e^x + 1}{e^x + 1}}{1} = 2$$

$$\ln \lim_{x\to 0^+} y = 2 \Rightarrow \lim_{x\to 0^+} y = e^2$$

$$\therefore \lim_{x\to 0^+} (e^x + x)^{\frac{1}{x}} = e^2$$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
من اعداد أ.د نزار عبد الجليل

Matrices and Determents

Def: Any arrangement of the elements in the following form

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

is called a matrix of order $m \times n$

where m = number of rows.

and n = number of columns.

Equal matrices:

$$\begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \\ 4 & -5 & 6 \end{bmatrix} \leftarrow \quad 3 \times 3 \text{ matrix}$$

iff $a = 2$, $b = 1$, $c = 3$, $e = 1$, $f = 0$, $g = -2$
$h = 4$, $i = -5$, $j = 6$.

Addition and Substraction:

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ 0 & 8 \end{bmatrix} + B = \begin{bmatrix} 4 & 5 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 4 & 9 \\ 3 & 10 \end{bmatrix}$

Note $[A]$ and $[B]$ must be of the same order

Multiplication by constant.

Def: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $k = z \Rightarrow k.A = \begin{bmatrix} az & bz \\ cz & dz \end{bmatrix}$

Let $[A]$, $[B]$ be two matrices such that number of rows of $[B]$ = number of columns of $[A]$

Then $[A]_{m \times n} \cdot [B]_{n \times k} = [C]_{m \times k}$.

Ex. Find $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times1}$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+3+6 \\ 0+4+15 \end{bmatrix} = \begin{bmatrix} 11 \\ 19 \end{bmatrix}.$$

Types of Matrices:

1. Square Matrix, if, $m=n$, $Ex = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 4 \\ -1 & 0 & 3 \end{bmatrix}$

2. Row matrix, if, $m=1$, Ex: $[A] = \begin{bmatrix} 3 & -1 & 2 & 7 \end{bmatrix}$

3. Column matrix, if, $n=1$, $Ex = [A] = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}_{3\times1}$

4. Identity matrix, if sq. matrix with unity main diagonal

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note in general $A \cdot B \neq B \cdot A$.

6. Symmetric matrix $[A] = \begin{bmatrix} 2 & 3 & -5 \\ 3 & 1 & 2 \\ -5 & 2 & -1 \end{bmatrix}$, is a square matrix about main diagonal.

7. Skew symmetric matrix: is a square matrix, and the main diagonal are zero

$$Ex = \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 2 \\ 5 & -2 & 0 \end{bmatrix}$$

8. Upper triangular matrix: is a square matrix with elements zero below the main diagonal:

Ex: $[A] = \begin{bmatrix} 1 & 4 & -6 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

9. Lower triangular matrix:-
is a sequare matrix with elements zero above the main diagonal

Ex: $[A] = \begin{bmatrix} 3 & 0 & 0 \\ 7 & 2 & 0 \\ 1 & 4 & -1 \end{bmatrix}$

10. Transpose of a matrix:

Let $[A] = \underset{3\times 2}{\begin{bmatrix} 3 & 5 \\ 6 & 4 \\ -2 & 1 \end{bmatrix}} \Rightarrow A^T = \begin{bmatrix} 3 & 6 & -2 \\ 5 & 4 & 1 \end{bmatrix}_{2\times 3}$

system of simultaneous Linear equations in a matrix form:

$$2x_1 - 3x_2 + 8x_3 = 6$$
$$x_1 + 7x_2 = -4$$
$$5x_1 + x_2 - x_3 = 5$$

in matrix form:

$$\begin{bmatrix} 2 & -3 & 8 \\ 1 & 7 & 0 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 5 \end{bmatrix}$$

$$A \cdot X = B$$

11: Symmetric matrix: Ex: $\cancel{\begin{bmatrix} 7 & 2 & -1 \\ 2 & 5 & 4 \\ -1 & 4 & 3 \end{bmatrix}}$

Note that $A = A^T$

12. Diagonal matrix: is a sequare matrix with all elements zero expect on the Leading diagonal $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

13 Unit matrix. is a sequare matrix in which the elements on the Leading diagonal are one, The unit matrix is denoted by 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14: Null Matrix: is a matrix whose elements are all zero.

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverse of a square matrix

Ex: Find A^{-1}, if $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix}$.

Sol
$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{vmatrix} = 1 \cdot [-20+6] - 2 \cdot [15+10] + 1[9+20]$$
$$= -14 - 50 + 29 = -35$$

$$A_{11} = + \quad -4 \times 5 - (-2 \times 3) = -14$$
$$A_{12} = -[3 \times 5 - (-2 \times 5)] = -25$$
$$A_{13} = [2 \times 5 - (-3 \times 1)] = -7$$
$$\vdots$$

$$\text{cof } A = \begin{bmatrix} -14 & -25 & 29 \\ -7 & 0 & 7 \\ 0 & 5 & -10 \end{bmatrix}$$

$$\text{adj} = (\text{cof} A)^T = \begin{bmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-35} \begin{bmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{bmatrix}$$

$$X = A^{-1} \cdot B = \frac{-1}{35} \begin{bmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$X = \frac{-1}{35} \begin{bmatrix} -70 \\ -105 \\ 140 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \Rightarrow x = 2, \ y = 3, \ z = -4.$$

H.W Sole the following system.

$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1 + 3x_2 - x_3 = 11$$
$$2x_1 - 2x_2 + 5x_3 = 3$$

Ans. $x_1 = -1$, $x_2 = 5$, $x_3 = 3$

Determinants

if A is a square matrix, then det A or $|A|$ is a number calculated from A and found as follows.

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$

Ex: Let $A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \Rightarrow |A| = 3\times -2 - 1\times 4 = -10$

Minors: is a determinant of order $n-1$ obtained from A after deleting the ith row and jth column.

Ex: Let $[A] = \begin{bmatrix} 1 & 2 & -4 \\ 2 & -5 & 6 \\ 13 & 6 & 7 \end{bmatrix}$

The minor of $a_{11} = \begin{bmatrix} -5 & 6 \\ 6 & 7 \end{bmatrix}$

To Find det A, Find the cofactor of any row or any column that has maximum zero and malt and sum. Note for $n=3$, only quick calculation of det A.

Ex: let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$ then.

$$\left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right. = 45 + 84 + 96 - 105 - (-48) - (-72)$$

$$= 240.$$

Properties of Determinants:

1. if two rows (or columns) are identical then $|A| = 0$ Ex. $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 7 \\ -1 & 4 & 7 \end{vmatrix} = 0$

2. Interchanging any two rows or columns changes the sigh of the determinants.

$$E = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & 5 & 7 \end{vmatrix} = - \begin{vmatrix} 3 & 1 & 0 \\ 2 & -1 & 3 \\ 2 & 5 & 7 \end{vmatrix}$$

3. $|A^T| = |A|$

Ex:
$$\begin{vmatrix} 2 & -4 & 18 \\ 1 & 0 & 6 \\ 5 & -1 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 & 9 \\ 1 & 0 & 6 \\ 5 & -1 & 3 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 1 & -2 & 3 \\ 1 & 0 & 2 \\ 5 & -1 & 1 \end{vmatrix}$$

$$\therefore \; |AB| = |A| \cdot |B|$$

Grammer's Rule method

Solve the following system of Linear eqs. using Grammer's Rule.

$$-2x + 3y - z = 1$$
$$x + 2y - z = 4$$
$$-2x - y + z = -3$$

Sol
$$\begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

$$A \cdot X = B.$$

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2 \neq 0.$$

$$x = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{-2} = \frac{-4}{-2} = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3$$

$$Z = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{-2} = \frac{-8}{-2} = 4$$

Adj of a matrix

Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$ find adj A.

Sol adj $A = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & -28 \end{bmatrix}$

Inverse of a Square Matrix

if A is $n \times n$ matrix and $|A| \neq 0$ then

$A^{-1} = \dfrac{\text{adj } A}{|A|}$, to find A^{-1}

1. Find $|A| \neq 0$.
2. Find cof A
3. Find adj $A = [\text{cof } A]^T$
4. Find $A^{-1} = \dfrac{\text{adj } A}{|A|}$

Ex: Find A^{-1} if $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

Sol 1. $|A| = -94 \neq 0$.

2. cof $A = \begin{bmatrix} -18 & 17 & -6 \\ -6 & -10 & -2 \\ -10 & -1 & 28 \end{bmatrix}$

3. adj $A = [\text{cof } A]^T = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$

4: $A^{-1} = \dfrac{adj}{|A|} = \dfrac{1}{-94}\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$

To check $A \cdot A^{-1} = A^{-1} \cdot A = I$. H.W.

$\boxed{\text{Solution of set of L.E. by using inverse matrix method}}$

<u>Ex</u> Solve the system of L.E using inverse Matrix Method.

$$x - 2y = 1$$
$$2x + 3y + z = 7$$
$$-x + 2z = 8$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix}$$

1. $|A| = 16 \neq 0$ check.

2. cof A. $A_{11} = 6$, $A_{12} = -5$, $A_{13} = 3$
$A_{21} = 4$, $A_{22} = 2$, $A_{23} = -(-2) = 2$
$A_{31} = +(-2)$, $A_{32}(-1)$, $A_{33} = (+)7$.

$$\text{cof } A = \begin{bmatrix} 6 & -5 & 3 \\ 4 & 2 & 2 \\ -2 & -1 & 7 \end{bmatrix}$$

3. $adj\, A = [\text{cof } A]^T = \begin{bmatrix} 6 & 4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & 7 \end{bmatrix}$

4. $A^{-1} = \dfrac{adj\, A}{|A|} = \dfrac{1}{16}\begin{bmatrix} 6 & 4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & 7 \end{bmatrix}$

$$X = [A^{-1}] \cdot [B]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 6 & 4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix}$$

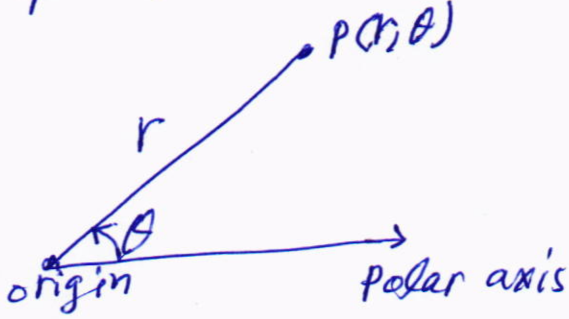
$\therefore x = \frac{9}{8}$, $y = \frac{1}{16}$, $z = \frac{73}{16}$

Note: to Find A^{-1} for

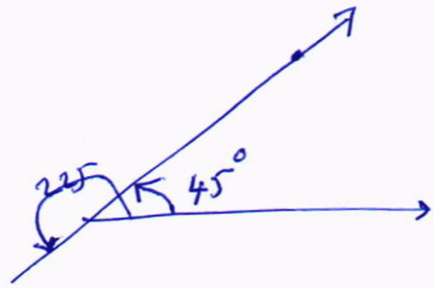
$$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Polar Coordinates.

The point $p(r,\theta)$ can be represent. as:

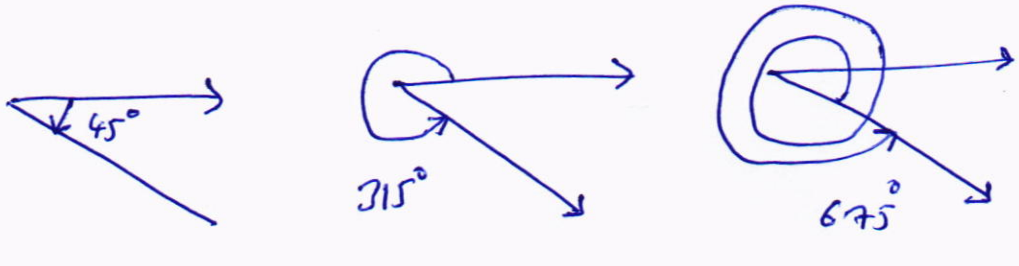


Ex:



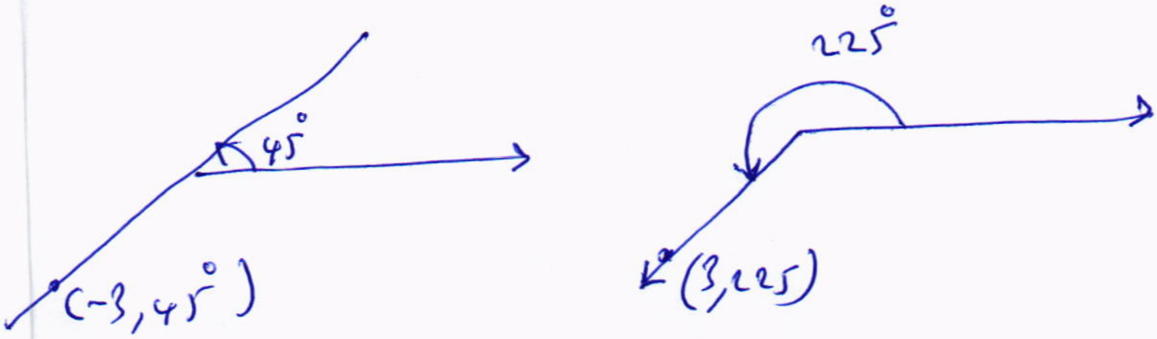
note 1: $(r, \theta + n360^\circ) = (r, \theta - n360^\circ) = (r, \theta)$

Ex: $(1, 45^\circ) = (1, 315^\circ) = (1, 675^\circ)$



note 2: $(-r, \theta) = (r, \theta + 180^\circ)$.

Ex: $(3, 45^\circ) = (3, 225^\circ)$



محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
م.د. أ.د. نزار عبد الجليل

Change from rectangular coo. into polar coo. and conversely

$(x,y) \rightleftarrows (r,\theta)$ using.

$x = r\cos\theta$ $\Rightarrow$ $r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\theta = \tan^{-1}\frac{y}{x}$

<u>Ex:</u> Find the rectangular coo of $(6, 135^\circ)$.

<u>Sol</u> $x = r\cos\theta = 6\cos(135^\circ) = 6\left(\frac{-\sqrt{2}}{2}\right) = -3\sqrt{2}$

$y = r\sin\theta = 6\sin(135^\circ) = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$

$\therefore$ the point is $(-3\sqrt{2}, 3\sqrt{2})$.

<u>Ex:</u> Find the polar coo of $(-2, 2\sqrt{3})$

$r = \sqrt{x^2+y^2} \Rightarrow \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2\sqrt{3}}{-2} = \tan^{-1} -\sqrt{3} = \frac{2\pi}{3}$

Since $(-2, 2\sqrt{3})$ lies on the second quadrent

$\therefore$ the point is $\left(4, \frac{2\pi}{3}\right)$.

<u>H.W</u>:

1) Find the rectangular coo of the points

1) $\left(6, \frac{\pi}{6}\right)$, 2. $\left(7, \frac{2\pi}{3}\right)$, 3) $\left(8, \frac{9\pi}{4}\right)$, 4) $(0, \pi)$

2) Find the polar coo of the points,

1) $(2\sqrt{3}, -2)$, 2) $(0, -2)$, 3) $(1, 1)$, 4) $(-3, 3\sqrt{2})$

<u>Ex:</u> change $r = 2\sin\theta$ into recantgular coo.

<u>Sol</u> $r = 2\sin\theta$ $\times r$

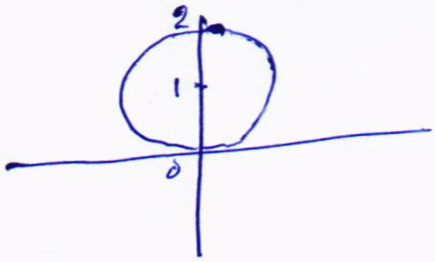
$r^2 = 2r\sin\theta$

$x^2 + y^2 = 2y$

$x^2 + y^2 - 2y = 0.$

$x^2 + y^2 - 2y + 1 - 1 = 0$

$x^2 + (y-1)^2 = 1$



which is circle of radius $= 1$
and center $= (0, 1)$.

Ex. Change $r = 5\sec\theta$ into rectangular coo.

Sol $r = 5\sec\theta$

$r = 5 \cdot \dfrac{1}{\cos\theta}$

$r\cos\theta = 5$

$x = 5$

Ex. Express the equation $2x - 5y = 3$ in polar coo.

Sol $2x - 5y = 3$

$2r\cos\theta - 5r\sin\theta = 3$

$r(2\cos\theta - 5\sin\theta) = 3$

$\therefore r = \dfrac{3}{2\cos\theta - 5\sin\theta}$

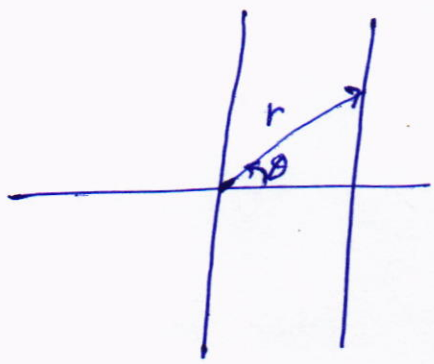
Graphs in polar coo

1) lines $x = a$

$r\cos\theta = a$

$r = \dfrac{a}{\cos\theta}$

$r = a\sec\theta.$

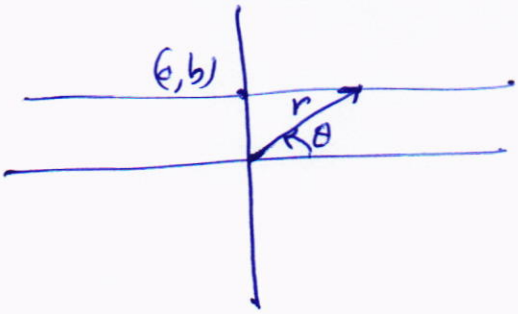


$$y = b$$

$$r\sin\theta = b$$

$$r = \frac{b}{\sin\theta}$$

$$r = b\csc\theta$$



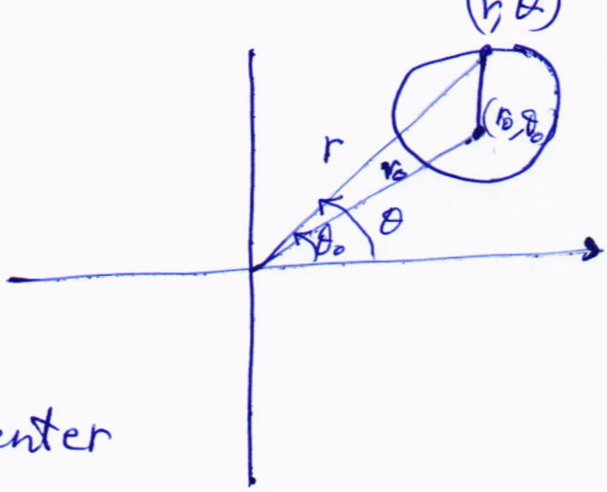
The general eq of the Line is

$$Ax + By + c = 0$$

$$Ar\cos\theta + Br\sin\theta = -c$$

$$r(A\cos\theta + B\sin\theta) = -c$$

$$\therefore r = \frac{-c}{A\cos\theta + B\sin\theta}$$



$\boxed{\text{Circles in polar coo}}$

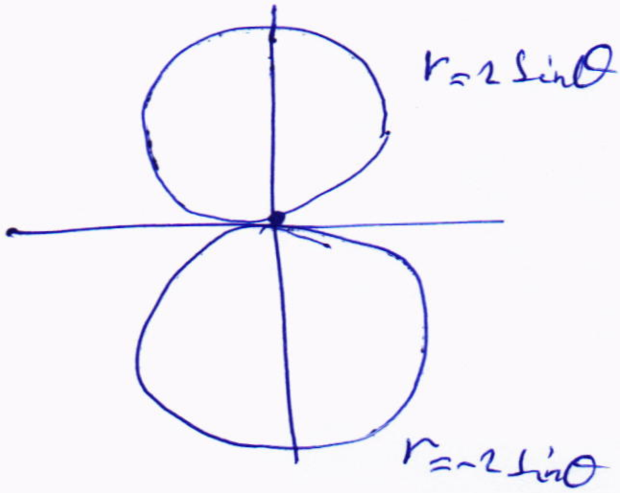
$$r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0) = a^2$$

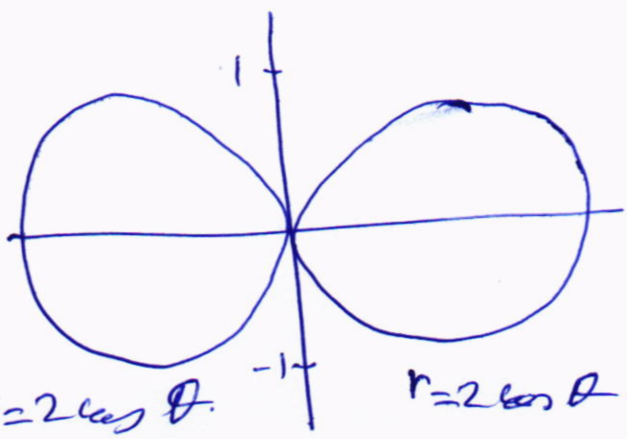
represent a circle of center (r, θ_0) and $r = a$.

if the center $(a, 0)$ or $r_0 = a$, $\theta_0 = 0$

$$r^2 - 2ar\cos\theta = 0$$

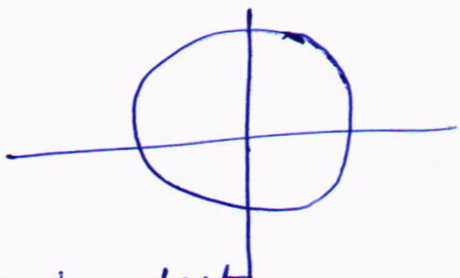
$$r = 2a\cos\theta$$





if the center (0,0) or $r_0 = 0$, $\theta_0 = 0$

$r = a$, $a = 2$



Symmetric test:

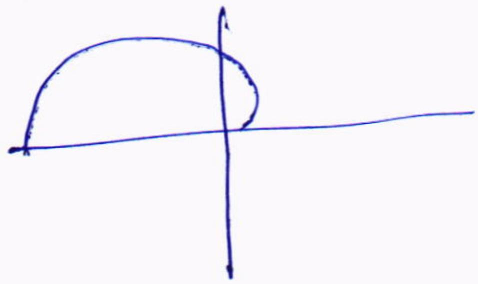
1) The curve $r = f(\theta)$ symmetric about y-axis, if θ replaced by $-\theta$ and the eq equivalent.

2) The curve $r = f(\theta)$ symmetric about x-axis if θ replaced by $\pi - \theta$ and the eq equivalent

3) The curve $r = f(\theta)$ is symmetric about the origin if r replaced by $-r$, and the eq - equivalent.

Ex: sketch the curve $r = a(1 - \cos\theta)$ cardioids

it is symm about x-axis, so we draw from $\theta = 0$ to $\theta = \pi$

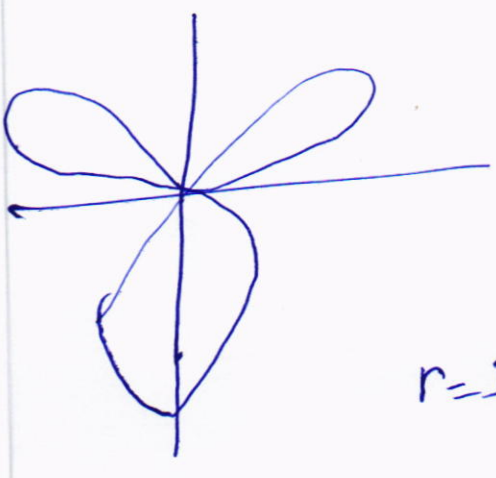


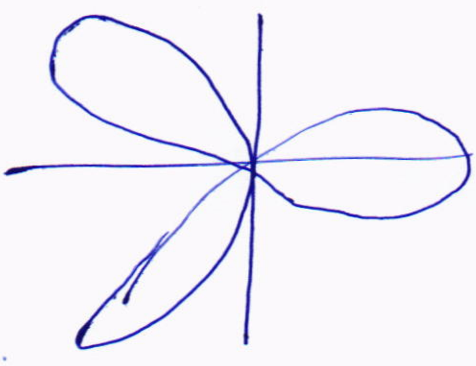
Ex: sketch the curve

$r = 5 \sin 3\theta$.

Sol if $n = 3$ odd.

numbe of petals $= 3$





$r = 5 \cos 3\theta$.

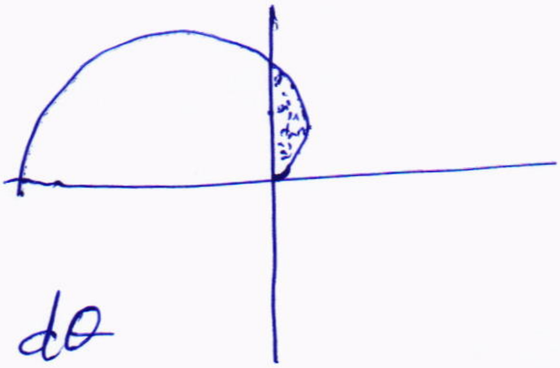
Area in polar coo

$$A = \int_{\alpha}^{\beta} \tfrac{1}{2}\left[f(x)\right]^2 d\theta = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta .$$

__Ex.:__ Find the area of the region in the first qudrant ~~wh~~ within the cardioid.

$$r = 1 - \cos\theta .$$

__Sol__



$$A = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \tfrac{1}{2}\int_{\alpha}^{\beta} (1-\cos\theta)^2\, d\theta$$

$$= \tfrac{1}{2}\int_{0}^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \tfrac{1}{2}\int_{0}^{\pi/2} (1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$

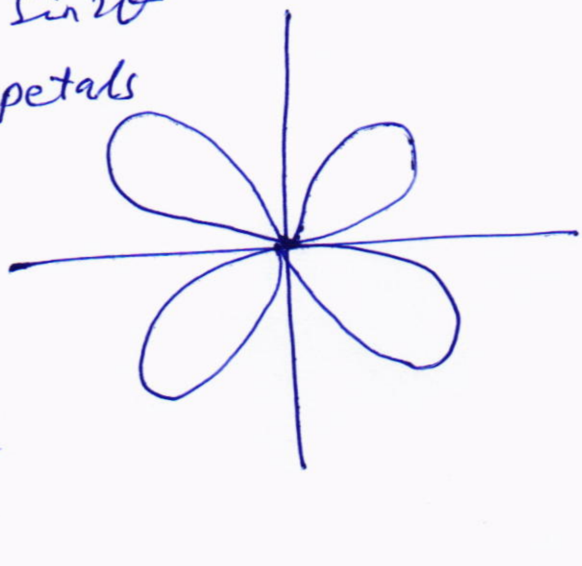
$$= \tfrac{1}{2}\int_{0}^{\pi/2} (\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{3}{8}\pi - 1$$

__Ex.:__ Find the area of the region enclosed by the rose curve $r = \sin 2\theta$

__Sol__ From symmetry of the petals



$$A = 4\int_{0}^{\pi/2} \tfrac{1}{2} r^2\, d\theta .$$

$$= 2\int_{0}^{\pi/2} [\sin 2\theta]^2\, d\theta$$

$$= 2\int_{0}^{\pi/2} \tfrac{1}{2}(1-\cos 4\theta)\, d\theta$$

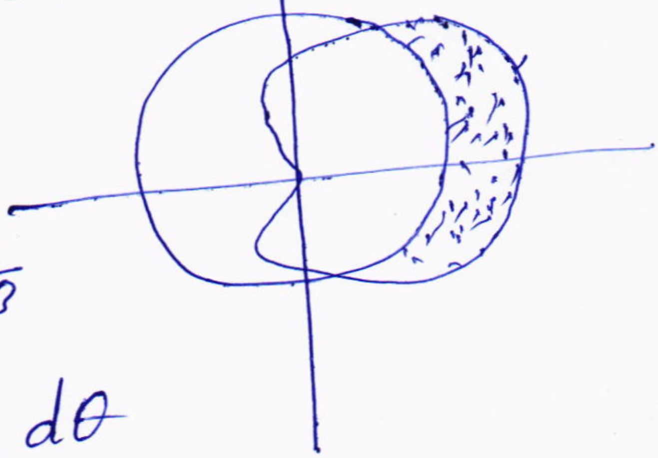
$= \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi}{2}$

Area between two polar curves

if $r_1 = f_1(\theta)$, $r_2 = f_2(\theta)$ then.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[r_1^2 - r_2^2\right] d\theta.$$

Ex.: Find the area of the region that is inside the cardioid $r = 4 + 4\cos\theta$ and outside $r = 6$.



Sol $6 = 4 + 4\cos\theta$

$2 = 4\cos\theta$.

$\cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$, $\theta = \frac{\pi}{3}$

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[r_1^2 - r_2^2\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[4 + 4\cos\theta\right]^2 - 6^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[16 + 32\cos\theta + 16\cos^2\theta - 36\right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left[16\cos\theta + 8\cos^2\theta - 10\right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left\{16\cos\theta + 8\left[\frac{1}{2}(1 + \cos 2\theta)\right] - 10\right\} d\theta$$

$$= \left[16\sin\theta + 4\theta + 2\sin 2\theta - 10\theta\right]_{-\pi/3}^{\pi/3} = 18\sqrt{3} - 4\pi$$

to Find the points of the intersection

<u>Ex:</u> $r = 1 - \cos\theta$.
$r = 1 + \cos\theta$.

<u>Sol</u> $1 - \cos\theta = 1 + \cos\theta$.

$2\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}$

and $r = 1$

$\therefore$ the points of intersection are $(1, \frac{\pi}{2}), (1, \frac{3\pi}{2})$.

Arc Length of polar curves

if $r = f(\theta)$ is a polar curve such that $f'(\theta)$ is continuous for

$$\alpha \le \theta \le \beta$$

then the arc Length defined by.

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

<u>Ex:</u> Find the arc Length of the polar curve $r = e^{\theta}$ from $\theta = 0$, $\theta = 1$

<u>Sol</u> $\because r = e^{\theta} \Rightarrow \frac{dr}{d\theta} = e^{\theta}$

$$\therefore L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$$L = \int_0^1 \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^1 \sqrt{2e^{2\theta}}$$

$$= \sqrt{2}\int_0^1 e^{\theta}\, d\theta = \sqrt{2}\left[e^{\theta}\right]_0^1 = \sqrt{2}(e-1)$$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
مدرس المادة أ.د. نور وليد الليل

The Complex numbers

The complex number $Z = x + iy$, $i = \sqrt{-1}$
where x is a real number and y is an imajenery number

1) addition and subtraction.

Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

then $z_1 + z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$.

2) Multiplication two complex numbers.

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Ex. let $z_1 = 3 + 2i$

$z_2 = -4 + 5i$

then $z_1 + z_2 = [3 + (-4)] + i[(2) + 5] = -1 + 7i$

$z_1 - z_2 = [3 - (-4)] + i[2 - 5] = 7 - 3i$

$2 \cdot z_1 = 2(3 + 2i) = 6 + 4i$

$$z_1 \cdot z_2 = (3 + 2i) + i(-4 + 5i)$$

$$= [(3 \times -4) - 2 \times 5] + i[3 \times 5 + 2 \times -4].$$

$$= [-12 - 10] + i[15 - 8] = -22 + 7i$$

The conjugate of a complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$

Properties:

1) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

2) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

3) $z \cdot \bar{z} = x^2 + y^2$

4) $z + \bar{z} = 2x$

5) $z - \bar{z} = 2iy$

Ex. Find : $\frac{1+5i}{3+2i}$

Sol $\frac{(1+5i)}{(3+2i)} \cdot \frac{(3-2i)}{(3-2i)}$

$= \frac{[1\times 3 - 5\times -2 + i(1\times -2) + 5\times 3]}{9+4}$

$= \frac{13+13i}{13} = \frac{13(1+i)}{13} = 1+i$

The magnitude and argument of Z

Each complex number Z can be written in polar coo. as $Z=(r,\theta)$, where.

$r = |Z| = \sqrt{x^2+y^2}$ $\qquad \theta = \arg Z = \tan^{-1}\frac{y}{x}$

Ex. The complex number $Z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ can be written in polar form.

$r = |Z| = \sqrt{x^2+y^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \tan^{-1}\frac{1}{\sqrt{3}} \Longrightarrow \theta = \frac{\pi}{6}$

$\therefore (r,\theta) = \left(1, \frac{\pi}{6}\right).$

$Z = x+iy = r\cos\theta + ir\sin\theta$
$= r(\cos\theta + i\sin\theta)$

$Z = re^{i\theta}.$

where $\boxed{e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{Euler formula}}$

if $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta_2}$ then

$$Z_1 \cdot Z_2 = r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \, e^{i(\theta_1 - \theta_2)}$$

Ex: let $Z_1 = 6\, e^{i\pi/2}$, $Z_2 = 3\, e^{i\pi/6}$

Find $Z_1 \cdot Z_2$, $\frac{Z_1}{Z_2}$

Sol $Z_1 Z_2 = 6 \times 3 \cdot e^{i\left(\frac{\pi}{2} + \frac{\pi}{6}\right)}$

$= 18 \, e^{i\,2\pi/3}$

$= 18\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 18\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -9 + 9\sqrt{3}\, i$

$\frac{Z_1}{Z_2} = \frac{6}{3} \, e^{i\left(\frac{\pi}{2} - \frac{\pi}{6}\right)}$

$= 2\, e^{i\pi/3}$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 1 + \sqrt{3}\, i$

$\boxed{\text{NOTE} \quad \because \left(e^{i\theta}\right)^n = e^{in\theta}}$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

This is called De Moiver's Law

Ex: Use De Moiver's Law ~~to~~ find $\cos 4\theta$ $\sin 4\theta$.

Sol $\because \quad n = 4$

محاضرات في الرياضيات
كلية الهندسة / المرحلة الأولى
مساعد أ.د. نزار عبد الجليل

Roots of complex numbers.

$$(z)^{p/q} = r^{p/q}\left(e^{i\left(\frac{p}{q}(\theta+2k\pi)\right)}\right) \qquad k=0,1,2,\dots,q-1$$

$$= r^{p/q}\left(\cos\frac{p}{q}(\theta+2k\pi) + i\sin\frac{p}{q}(\theta+2k\pi)\right).$$

Ex: Find the roots of $\sqrt[4]{-8i} = (-8i)^{1/4}$

$$z = x+iy = 0-8i \quad , \quad x=0 \quad , \quad y=-8$$

$$r = |z| = \sqrt{x^2+y^2} = \sqrt{0+(-8)^2} = 8$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-8}{0} = -\frac{\pi}{2} = \frac{3\pi}{2}$$

$$z^{p/q} = r^{p/q}\left(\cos\frac{p}{q}(\theta+2k\pi) + i\sin\frac{p}{q}(\theta+2k\pi)\right) \quad k=0,1,2,3$$

$$z_0 = 8^{1/4}\left(\cos\frac{1}{4}\left(\frac{3\pi}{2}\right) + i\sin\frac{1}{4}\left(\frac{3\pi}{2}\right)\right) \quad k=0$$

$$z_1 = 8^{1/4}\left(\cos\frac{1}{4}\left(\frac{3\pi}{2}+2\pi\right) + i\sin\frac{1}{4}\left(\frac{3\pi}{2}+2\pi\right)\right) \quad k=1$$

$$z_2 = 8^{1/4}\left(\cos\frac{1}{4}\left(\frac{3\pi}{2}+4\pi\right) + i\sin\frac{1}{4}\left(\frac{3\pi}{2}+4\pi\right)\right) \quad k=2$$

$$z_3 = 8^{1/4}\left(\cos\frac{1}{4}\left(\frac{3\pi}{2}+6\pi\right) + i\sin\frac{1}{4}\left(\frac{3\pi}{2}+6\pi\right)\right) \quad k=3$$

H.W Find the Root of $(-1-i)^{4/5}$

Complex Functions

1) $e^{z} = e^{x+iy} = e^{x}\cdot e^{iy} = e^{x}(\cos y + i\sin y)$

$= e^{x}(\cos y + i\, e^{x}\sin y)$

2) $\ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$

$= \ln r + i\theta = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

3) $\sin z = \sin(x+iy)$

$= \sin x \cos iy + \sin iy \cos x$

$= \sin x \cosh y + i \sinh y \cos x$

محاضرات في الرياضيات
م.م. إعداد أ.د. نزار دنبد الحلبي
كلية الهندسة المرحلة الأولى

H.W.: 1) Determine whether f and g are inverse Functions.

a) $f(x) = 4x$, $g(x) = \frac{1}{4}x$ Ans: Yes.

b) $f(x) = x^4$, $g(x) = \sqrt[4]{x}$ Ans No.

2) Solve for y if $\ln(y-1) - \ln y = 2x$

Ans: $y = \frac{1}{1-e^{2x}}$

3) Find $\frac{dy}{dx}$ of $x^2 + \sin 2x$ Ans $e^{x^2+\sin 2x}(2x + 2\cos 2x)$

4) if $\sinh x = \frac{-3}{4}$, Find the value of the:

$\tanh x = \frac{\sinh x}{\cosh x}$ Ans $\frac{-3}{4}$

5) Find y', if $y = \sinh^{-1}\sqrt{x}$ Ans: $\frac{1}{2\sqrt{x(1+x)}}$

6) Evaluate $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx$ Ans: $12\sinh\left(\frac{x}{2} - \ln 3\right)$

7) Evaluate the integrals in terms of:

a) inverse hyperbolic Function

b) natural logarithms.

of: $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$ Ans a) $\sinh^{-1}(\sqrt{3})$ b) $\ln(\sqrt{3}+2)$

8) Evaluate $\int \ln x \, dx$ Ans $x\ln x - \int dx = x\ln x - x + c$

9) Evaluate $\int \frac{dx}{2+2\sqrt{x}}$, Ans: $\sqrt{x} - \ln|1+\sqrt{x}| + c$

10) Find y', if $y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1}$, Ans: $\frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$

11) Use L'Hôpital's Rule $\lim_{x\to 0} \frac{3x - \sin x}{x}$ Ans. 2

12) Solve the Systems of equations by a matrix method

$$x_1 + 2x_2 + x_3 = 4$$
$$3x_1 - 4x_2 - 2x_3 = 2$$
$$5x_1 + 3x_2 + 5x_3 = -1$$

Ans $x_1 = 2$, $x_2 = 3$, $x_3 = -4$