2.15 KIRCHHOFF'S LAWS

Ohm's law by itself is not sufficient to analyze circuits. However, when it is coupled with Kirchhoff's two laws, we have a sufficient, powerful set of tools for analyzing a large variety of electric circuits. Kirchhoff's laws were first introduced in 1847 by the German physicist Gustav Robert Kirchhoff (1824–1887). These laws are formally known as Kirchhoff's current law (**KCL**) and Kirchhoff's voltage law (**KVL**).

2.15.1 Kirchhoff's current law

Kirchhoff's first law is based on the law of conservation of charge, which requires that the algebraic sum of charges within a system cannot change.

Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.

Mathematically, **KCL** implies that

$$\sum_{n=1}^{N} i_n = 0 \tag{2.38}$$

where **N** is the number of branches connected to the node and in is the nth current entering (or leaving) the node. By this law, currents entering a node may be regarded as positive, while currents leaving the node may be taken as negative or vice versa. To prove **KCL**, assume a set of currents $i_k(t)$, k = 1, 2, . . . , flow into a node. The algebraic sum of currents at the node is

$$i_T(t) = i_1(t) + i_2(t) + i_3(t) + \cdots \tag{2.39}$$

Integrating both sides of **Eq. (2.39)** gives

$$q_T(t) = q_1(t) + q_2(t) + q_3(t) + \cdots \tag{2.40}$$

where $q_k(t) = \int i_k(t)\,dt$ and $q_T(t) = \int i_T(t)\,dt$. But the law of conservation of electric charge requires that the algebraic sum of electric charges at the node must not change; that is, the node stores no net charge. Thus $q_T(t) = 0 \rightarrow i_T(t) = 0$, confirming the validity of **KCL**.

Consider the node in **Fig. 2.33**. Applying KCL gives

$$\mathbf{i_1 + (-i_2) + i_3 + i_4 + (-i_5) = 0} \tag{2.41}$$

since currents $\mathbf{i_1}$, $\mathbf{i_3}$, and $\mathbf{i_4}$ are entering the node, while currents $\mathbf{i_2}$ and $\mathbf{i_5}$ are leaving it. By rearranging the terms, we get

$$\mathbf{i_1 + i_3 + i_4 = i_2 + i_5} \tag{2.42}$$

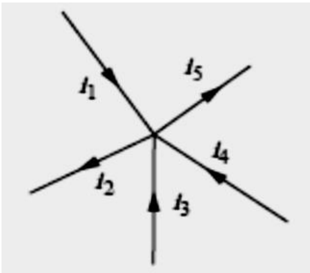


Figure 2.33 Currents at a node illustrating KCL.

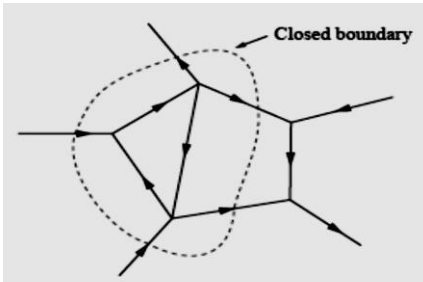


Figure 2.34 Applying KCL to a closed boundary.

Eq. (2.42) is an alternative form of **KCL**:

The sum of the currents entering a node is equal to the sum of the currents leaving the node.

Note that **KCL** also applies to a closed boundary. This may be regarded as a generalized case, because a node may be regarded as a closed surface shrunk to a point. In two dimensions, a closed boundary is the same as a closed path. As typically illustrated in the circuit of **Fig. 2.34**, the total current entering the closed surface is equal to the total current leaving the surface.

A simple application of **KCL** is combining current sources in parallel. The combined current is the algebraic sum of the current supplied by the individual sources. For example, the current sources shown in **Fig. 2.35(a)** can be combined as in **Fig. 2.35(b)**. The combined or equivalent current source can be found by applying **KCL** to node **a**.

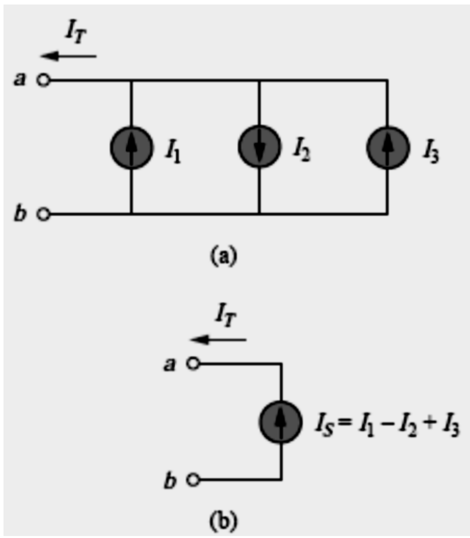


Figure 2.35 Current sources in parallel: (a) original circuit, (b) equivalent circuit.

$$\mathbf{I_T + I_2 = I_1 + I_3}$$

or

$$\mathbf{I_T = I_1 - I_2 + I_3} \tag{2.43}$$

A circuit cannot contain two different currents, $\mathbf{I_1}$ and $\mathbf{I_2}$, in series, unless $\mathbf{I_1} = \mathbf{I_2}$; otherwise **KCL** will be violated.

2.15.2 Kirchhoff's voltage law

Kirchhoff's second law is based on the principle of conservation of energy:

Kirchhoff's voltage law (**KVL**) states that the algebraic sum of all voltages around a closed path (or loop) is zero.

Expressed mathematically, **KVL** states that

$$\sum_{m=1}^{M} v_m = 0 \tag{2.44}$$

Where **M** is the number of voltages in the loop (or the number of branches in the loop) and v_m is the mth voltage.

To illustrate **KVL**, consider the circuit in **Fig. 2.36**. The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop. We can start with any branch and go around the loop either clockwise or counterclockwise. Suppose we start with the voltage source and go clockwise around the loop as shown; then voltages would be $-v_1$, $+v_2$, $+v_3$, $-v_4$, and $+v_5$, in that order. For example, as we reach branch 3, the positive terminal is met first; hence we have $+v_3$. For branch 4, we reach the negative terminal first; hence, $-v_4$. Thus, **KVL** yields

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0 \tag{2.45}$$

Rearranging terms gives

$$v_2 + v_3 + v_5 = v_1 + v_4 \tag{2.46}$$

which may be interpreted as

Sum of voltage drops = Sum of voltage rises (2.47)

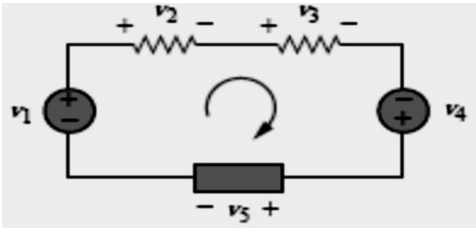


Figure 2.36 A single-loop circuit illustrating KVL.

This is an alternative form of **KVL**. Notice that if we had traveled counterclockwise, the result would have been $+v_1$, $-v_5$, $+v_4$, $-v_3$, and $-v_2$, which is the same as before, except that the signs are reversed. Hence, **Eqs. (2.45) and (2.46)** remain the same.

When voltage sources are connected in series, **KVL** can be applied to obtain the total voltage. The combined voltage is the algebraic sum of the voltages of the individual sources.

2.14.3 Steps to Apply Kirchhoff. Laws to Get Network Equations

The steps are stated based on the branch current method.

Step 1: Draw the circuit diagram from the given information and insert all the value of sources with appropriate polarities and all the resistances.

Step 2: Mark all the branch currents with assumed directions using **KCL** at various nodes and junction points. Kept the number of unknown currents as minimum as far as possible to limit the mathematical calculations required to solve them later on. Assumed directions may be wrong; in such case answer of such current will be mathematically negative which indicates the correct direction of the current.

Step 3: Mark all the polarities of voltage drops and rises as per directions of the assumed branch currents flowing through various branch resistance of the network. This is necessary for application of **KVL** to various closed loops.

Step 4: Apply **KVL** to different closed paths in the network and obtain the corresponding equations. Each equation must contain some element which is not considered in any preview equation.

2.16 Solving Simultaneous Equations and Cramer's Rule

Electric circuit analysis with the help of Kirchhoff's laws usually involves solution of two or three simultaneous equations. These equations can be solved by a systematic elimination of the variables but the procedure is often lengthy and laborious and hence more liable to error. Determinants and Cramer's rule provide a simple and straight method for solving network equations through manipulation of their coefficients. Of course, if the number of simultaneous equations happens to be very large, use of a digital computer can make the task easy. Let us assume that set of simultaneous equations obtained is, as follows,

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \dots\dots\dots\dots + a_{1n} x_n &= C_1 \\
a_{21} x_1 + a_{22} x_2 + \dots\dots\dots\dots + a_{2n} x_n &= C_2 \\
&\;\;. \\
&\;\;. \\
a_{n1} x_1 + a_{n2} x2 + \dots\dots\dots\dots + a_{nn} x_n &= C_n
\end{aligned}
$$

where C_1, C_2, ………, C_n constants. Then Cramer's rule says that form a system determinant **Δ** or **D** as,

$$\Delta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = D$$

Then obtain the subdeterminant **Dj** by replacing j^{th} column of **Δ** by the column of constants existing on right hand side of equations i.e. $C_1, C_2, \dots\, C_n$;

$$D_1 = \begin{bmatrix} C_1 & a_{12} & \cdots & a_{1n} \\ C_n & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ C_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} a_{11} & C_1 & \cdots & a_{1n} \\ a_{21} & C_2 & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & C_n & \cdots & a_{nn} \end{bmatrix}$$

and

$$D_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & C_1 \\ a_{21} & a_{22} & \cdots & C_2 \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & C_n \end{bmatrix}$$

The unknowns of the equations are given by Cramer's rule as,

$$X_1 = \frac{D_1}{D}, \quad X_2 = \frac{D_2}{D}, \cdots, X_n = \frac{D_n}{D}$$

Where $\mathbf{D_1}$, $\mathbf{D_2}$, …, $\mathbf{D_n}$ and **D** are values of the respective determents

Example 2.6: Apply Kirchhoff's laws to the circuit shown in figure 1 below

Indicate the various branch currents.

Write down the equations relating the various branch currents.

Solve these equations to find the values of these currents.

Is the sign of any of the calculated currents negative?

If yes, explain the significance of the negative sign.

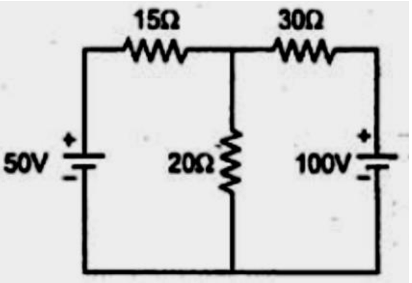


Figure 1

Solution: Application Kirchhoff's laws:

Step 1and 2: Draw the circuit with all the values which are same as the given network. Mark all the branch currents starting from **+ve** of any of the source, say +ve of 50 V source

Step 3: Mark all the polarities for different voltages across the resistance. This is combined with step 2 shown in the network below in Fig. 1 (a).

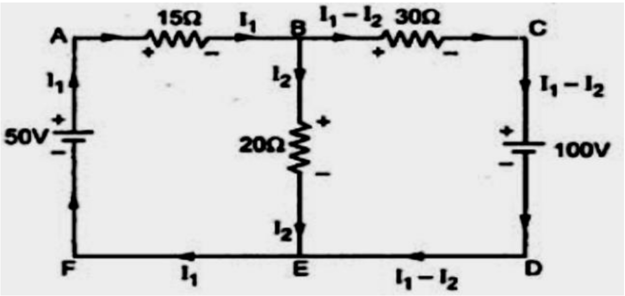


Figure 1 (a)

Step 4: Apply **KVL** to different loops.

Loop 1: A-B-E-F-A, $-15\,\mathbf{I_1} - 20\,\mathbf{I_2} + 50 = 0$

Loop 2: B-C-D-E-D, $-30\,(\mathbf{I_1} - \mathbf{I_2}) - 100 + 20\,\mathbf{I_2} = 0$

Rewriting all the equations, taking constants on one side,

$$\mathbf{15\,I_1 + 20\,I_2 = 50,} \qquad \mathbf{-30\,I_1 + 50\,I_2 = 100}$$

Apply Cramer's rule, $D = \begin{vmatrix} 15 & 20 \\ -30 & 50 \end{vmatrix} = 1350$

Calculating **D₁**, $D_1 = \begin{vmatrix} 50 & 20 \\ 100 & 50 \end{vmatrix} = 500$

$$I_1 = \frac{D_1}{D} = \frac{500}{1350} = 0.37\ A$$

Calculating **D₂**, $D_2 = \begin{vmatrix} 15 & 50 \\ -30 & 100 \end{vmatrix} = 3000$

$$I_2 = \frac{D_2}{D} = \frac{3000}{1350} = 2.22\ A$$

For $\mathbf{I_1}$ and $\mathbf{I_2}$ as answer is positive, assumed direction is correct.

: . For $\mathbf{I_1}$ answer is 0.37 A. For $\mathbf{I_2}$ answer is 2.22 A

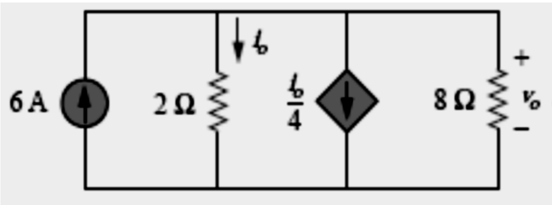
$$\mathbf{I_1 - I_2 = 0.37 - 2.22 = -1.85\ A}$$

Negative sign indicates assumed direction is wrong.

i.e. $\mathbf{I_1 - I_2}$ = 1.85 A flowing in opposite direction to that of the assumed direction.

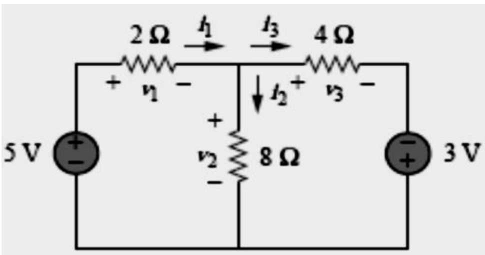
Practice problem 2.7: Find $\boldsymbol{v_o}$ and $\boldsymbol{i_o}$ in the circuit of Figure below.

Answer: 8 V, 4 A.



Practice problem 2 .8: Find the currents and voltages in the circuit shown below.

Answer: v_1 = 3 V, v_2 = 2 V, v_3 = 5 V, i_1 = 1.5 A, i_2 = 0.25 A, i_3 =1.25 A.



Methods of Analysis

3.1 INTRODUCTION

Having understood the fundamental laws of circuit theory (**Ohm's law** and **Kirchhoff's laws**), we are now prepared to apply these laws to develop two powerful techniques for circuit analysis: nodal analysis, which is based on a systematic application of Kirchhoff's current law (**KCL**), and mesh analysis, which is based on a systematic application of Kirchhoff's voltage law (**KVL**). The two techniques are so important that this chapter should be regarded as the most important in the lectures.

3.3 MESH ANALYSIS

Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variables. Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously. Recall that a loop is a closed path with no node passed more than once. A mesh is a loop that does not contain any other loop within it.

Nodal analysis applies **KCL** to find unknown voltages in a given circuit, while mesh analysis applies **KVL** to find unknown currents. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is planar. A **planar** circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is **nonplanar**. A circuit may have crossing branches and still be **planar** if it can be redrawn such that it has no crossing branches. For example, the circuit in **Fig. 3.11(a)** has two crossing branches, but it can be redrawn as in **Fig. 3.11(b)**. Hence, the circuit in **Fig. 3.11(a)** is planar. However, the circuit in **Fig. 3.12** is **nonplanar**, because there is no way to redraw it and avoid the branches crossing. Nonplanar circuits can be handled using nodal analysis, but they will not be considered in this text.

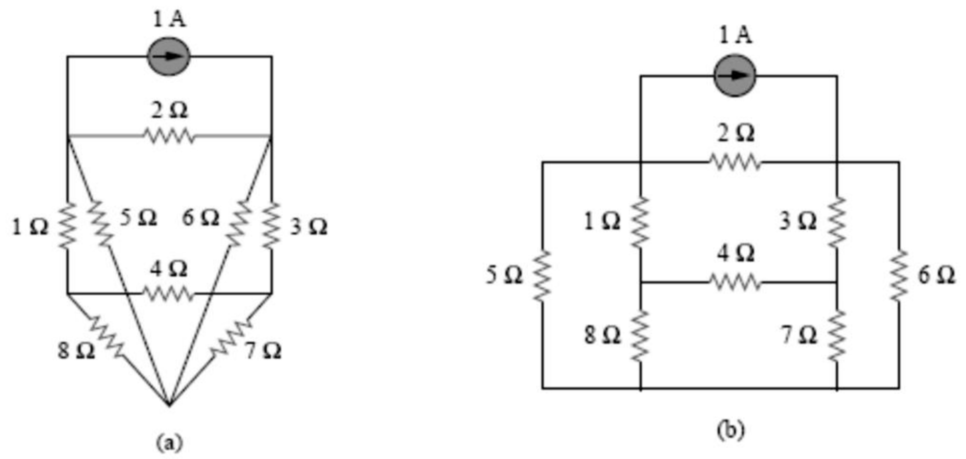


Figure 3.11 (a) A planar circuit with crossing branches, (b) the same circuit redrawn with no crossing branches.

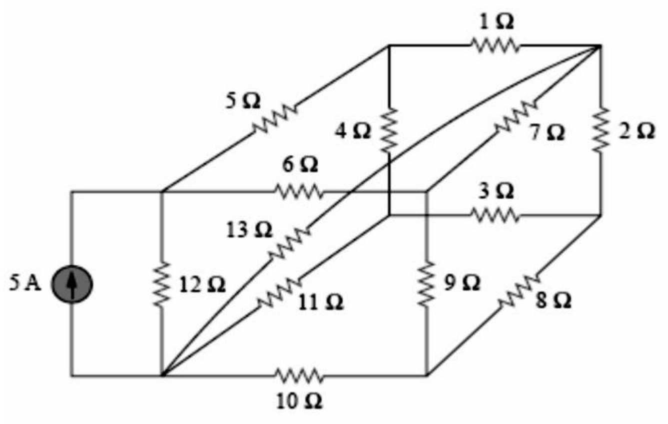


Figure 3.12 A nonplanar circuit.

To understand mesh analysis, we should first explain more about what we mean by a mesh.

A mesh is a loop which does not contain any other loops within it.

In **Fig. 3.13**, for example, paths **abefa** and **bcdeb** are meshes, but path **abcdefa** is not a mesh. The current through a mesh is known as mesh current. In mesh analysis, we are interested in applying **KVL** to find the mesh currents in a given circuit.

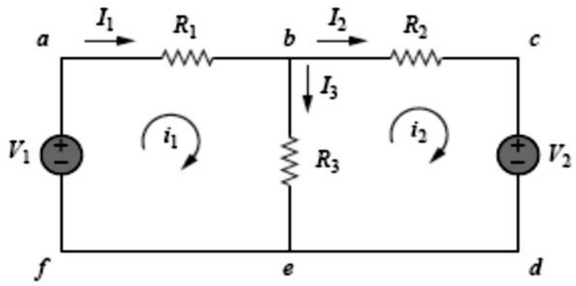


Figure 3.13 A circuit with two meshes.

In this section, we will apply mesh analysis to planar circuits that do not contain current sources. In the next sections, we will consider circuits with current sources. In the mesh analysis of a circuit with **n** meshes, we take the following three steps.

Steps to Determine mesh currents:

1. Assign mesh currents $\mathbf{i_1}$, $\mathbf{i_2}$, . . . , $\mathbf{i_n}$ to the n meshes.
2. Apply **KVL** to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting n simultaneous equations to get the mesh currents.

To illustrate the steps, consider the circuit in **Fig. 3.13**. The first step requires that mesh currents $\mathbf{i_1}$ and $\mathbf{i_2}$ are assigned to meshes 1 and 2. Although a mesh current may be assigned to each mesh in an arbitrary direction, it is conventional to assume that each mesh current flows clockwise.

As the second step, we apply **KVL** to each mesh. Applying **KVL** to mesh 1, we obtain

$$-V_1 + R_1 i_1 + R_3 (i_1 - i_2) = 0$$

or

$$(R_1 + R_3)\, i1 - R_3 i_2 = V_1 \qquad (3.13)$$

For mesh 2, applying KVL gives

$$R_2 i_2 + V_2 + R_3 (i_2 - i_1) = 0$$

or

$$-R_3 i_1 + (R_2 + R_3)\, i_2 = -V_2 \tag{3.14}$$

Note in **Eq. (3.13)** that the coefficient of i_1 is the sum of the resistances in the first mesh, while the coefficient of i_2 is the negative of the resistance common to meshes 1 and 2. Now observe that the same is true in **Eq. (3.14)**. This can serve as a shortcut way of writing the mesh equations.

The third step is to solve for the mesh currents. Putting **Eqs. (3.13)**. and **(3.14)** in matrix form yields

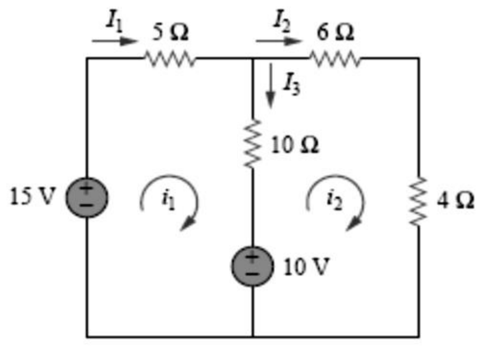
$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix} \tag{3.15}$$

which can be solved to obtain the mesh currents i_1 and i_2. We are at liberty to use any technique for solving the simultaneous equations. If a circuit has **n** nodes, **b** branches, and ***l*** independent loops or meshes, then $l = b - n + 1$. Hence, ***l*** independent simultaneous equations are required to solve the circuit using mesh analysis.

Notice that the branch currents are different from the mesh currents unless the mesh is isolated. To distinguish between the two types of currents, we use **i** for a mesh current and **I** for a branch current. The current elements I_1, I_2, and I_3 are algebraic sums of the mesh currents. It is evident from **Fig. 3.13** that

$$I_1 = i_1, \quad I_2 = i_2, \quad I_3 = i_1 - i_2 \tag{3.16}$$

Example 3.5: For the circuit in **Fig. 3.14**, find the branch currents I_1, I_2, and I_3 using mesh analysis.



Solution:

We first obtain the mesh currents using **KVL**. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

Figure 3.14 For Example 3.5.

$$\mathbf{3i_1 - 2i_2 = 1} \quad (3.5.1)$$

For mesh 2,

$$\mathbf{6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0}$$

or

$$\mathbf{i_1 = 2i_2 - 1} \quad (3.5.2)$$

Using the substitution method, we substitute **Eq. (3.5.2)** into **Eq. (3.5.1)**, and write

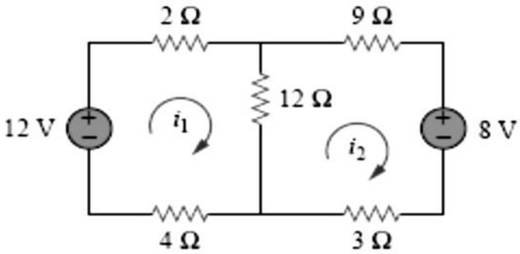
$$\mathbf{6i_2 - 3 - 2i_2 = 1 \Rightarrow \quad i_2 = 1\ A}$$

From **Eq. (3.5.2)**, $\mathbf{i_1 = 2i_2 - 1 = 2 - 1 = 1\ A}$. Thus,

$$\mathbf{I_1 = i_1 = 1\ A,\ I_2 = i_2 = 1\ A,\ I_3 = i_1 - i_2 = 0}$$

Practice problem 3.5: Calculate the mesh currents $\mathbf{i_1}$ and $\mathbf{i_2}$ in the circuit of Figure below.

Answer: i1 = 23 A, i2 = 0 A.



Example 3.6: Use mesh analysis to find the current $\mathbf{i_o}$ in the circuit in **Fig. 3.15**.

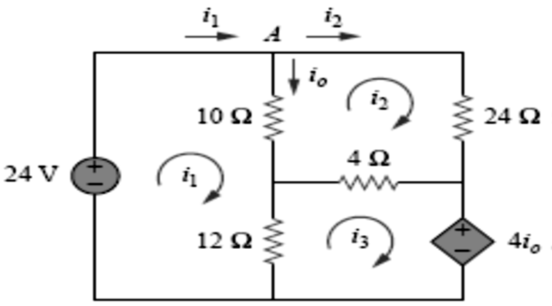


Figure 3.15 For Example 3.6.

Solution:

We apply **KVL** to the three meshes in turn. For mesh 1,

$$\mathbf{-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0}$$

or

$$\mathbf{11i_1 - 5i_2 - 6i_3 = 12} \tag{3.6.1}$$

For mesh 2,

$$\mathbf{24i_2 + 4(i_2 - i_3) + 10(i_2 - i_1) = 0}$$

or

$$\mathbf{-5i_1 + 19i_2 - 2i_3 = 0} \tag{3.6.2}$$

For mesh 3,

$$\mathbf{4i_o + 12(i_3 - i_1) + 4(i_3 - i_2) = 0}$$

But at node A, $\mathbf{i_o = i_1 - i_2}$, so that

$$\mathbf{4(i_1 - i_2) + 12(i_3 - i_1) + 4(i_3 - i_2) = 0}$$

or

$$\mathbf{-i_1 - i_2 + 2i_3 = 0} \tag{3.6.3}$$

In matrix form, **Eqs. (3.6.1)** to **(3.6.3)** become

$$\begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

We obtain the determinants as

$$D = \begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} = 192\ ,\ D_1 = \begin{bmatrix} 12 & -5 & -6 \\ 0 & 19 & -2 \\ 0 & -1 & 2 \end{bmatrix} = 432$$

$$D_2 = \begin{bmatrix} 11 & 12 & -6 \\ -5 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix} = 144\ ,\ D_3 = \begin{bmatrix} 11 & -5 & 12 \\ -5 & 19 & 0 \\ -1 & -1 & 0 \end{bmatrix} = 288$$

We calculate the mesh currents using **Cramer's rule** as

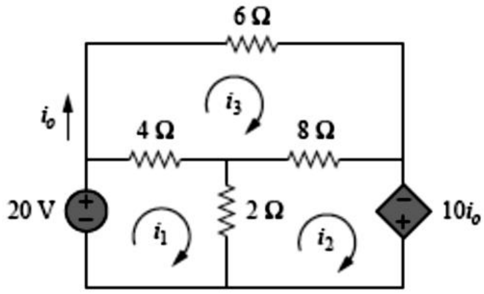
$$i_1 = \frac{D_1}{D} = \frac{432}{192} = 2.25\ A\ ,\ i_2 = \frac{D_2}{D} = \frac{144}{192} = 0.75\ A$$

$$i_3 = \frac{D_3}{D} = \frac{288}{192} = 1.5\ A$$

Thus, $\mathbf{i_o = i_1 - i_2 = 1.5\ A}$

Practice problem 3.6: Using mesh analysis, find $\mathbf{i_o}$ in the circuit in Figure below.

Answer: −5 A.



3.4 MESH ANALYSIS WITH CURRENT SOURCES

Applying mesh analysis to circuits containing current sources (dependent or independent) may appear complicated. But it is actually much easier than what we encountered in the previous section, because the presence of the current sources reduces the number of equations. Consider the following two possible cases.

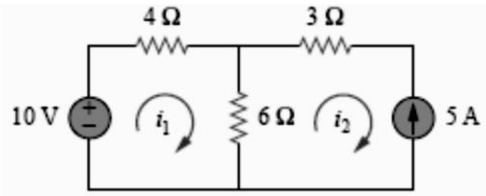


Figure 3.22 A circuit with a current source.

CASE 1: When a current source exists only in one mesh: Consider the circuit in **Fig. 3.16**, for example. We set $\mathbf{i_2} = -5$ A and write a mesh equation for the other mesh in the usual way, that is,

$$\mathbf{-10 + 4i_1 + 6(i_1 - i_2) = 0 \Rightarrow i_1 = -2\ A} \qquad (3.17)$$

CASE 2: When a current source exists between two meshes: Consider the circuit in **Fig. 3.17(a)**, for example. We create a supermesh by excluding the current source and any elements connected in series with it, as shown in **Fig. 3.17(b)**. Thus,

A supermesh results when two meshes have a (dependent or independent) current source in common.

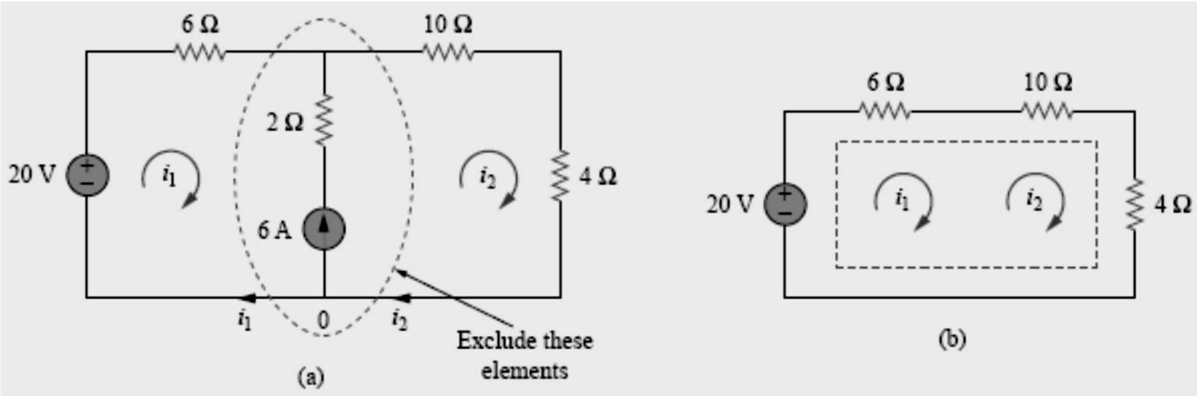


Figure 3.17 (a) Two meshes having a current source in common, (b) a supermesh, created by excluding the current source.

As shown in **Fig. 3.17(b)**, we create a supermesh as the periphery of the two meshes and treat it differently. (If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.) Why treat the supermesh differently? Because mesh analysis applies **KVL**—which requires that we know the voltage across each branch—and we do not know the voltage across a current source in advance. However, a supermesh must satisfy **KVL** like any other mesh.

Therefore, applying **KVL** to the supermesh in **Fig. 3.17(b)** gives

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

or

$$6i_1 + 14i_2 = 20 \qquad (3.18)$$

We apply **KCL** to a node in the branch where the two meshes intersect.

Applying **KCL** to node 0 in **Fig. 3.17(a)** gives

$$i_2 = i_1 + 6 \qquad (3.19)$$

Solving **Eqs. (3.18)** and **(3.19)**, we get

$$i_1 = -3.2\text{ A},\ i_2 = 2.8\text{ A} \qquad (3.20)$$

Note the following properties of a supermesh:

1. The current source in the supermesh is not completely ignored; it provides the constraint equation necessary to solve for the mesh currents.

2. A supermesh has no current of its own.

3. A supermesh requires the application of both **KVL** and **KCL**.

Example 3.7: For the circuit in **Fig. 3.18**, find $\mathbf{i_1}$ to $\mathbf{i_4}$ using mesh analysis.

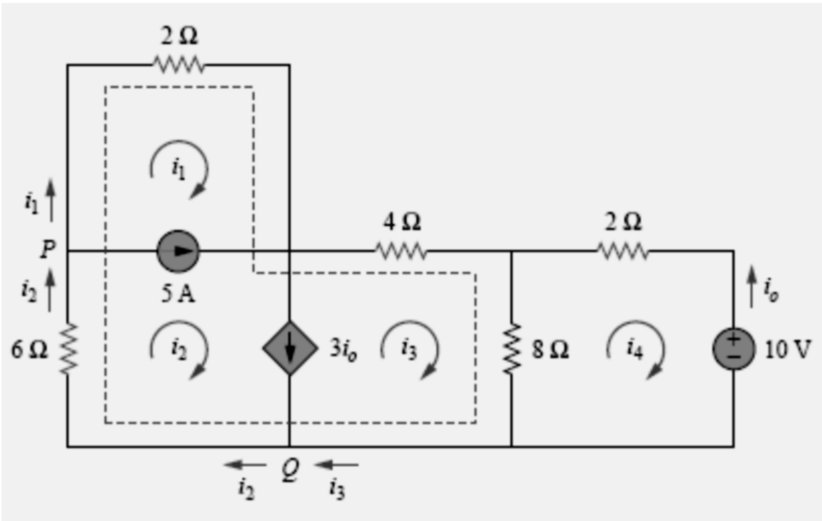


Figure 3.18 For Example 3.7.

Solution:

Note that meshes 1 and 2 form a supermesh since they have an independent current source in common. Also, meshes 2 and 3 form another supermesh because they have a dependent current source in common. The two supermeshes intersect and form a larger supermesh as shown. Applying **KVL** to the larger supermesh,

$$\mathbf{2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0}$$

or

$$\mathbf{i_1 + 3i_2 + 6i_3 - 4i_4 = 0} \qquad (3.7.1)$$

For the independent current source, we apply **KCL** to node P:

$$\mathbf{i_2 = i_1 + 5} \tag{3.7.2}$$

For the dependent current source, we apply KCL to node Q:

$$\mathbf{i_2 = i_3 + 3i_o}$$

But $\mathbf{i_o = -i_4}$, hence,

$$\mathbf{i_2 = i_3 - 3i_4} \tag{3.7.3}$$

Applying **KVL** in mesh 4,

$$\mathbf{2i_4 + 8(i_4 - i_3) + 10 = 0}$$

or

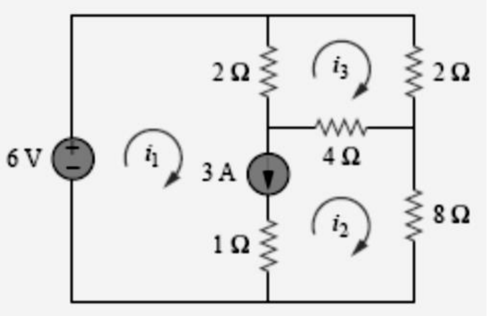
$$\mathbf{5i_4 - 4i_3 = -5} \tag{3.7.4}$$

From **Eqs. (3.7.1)** to **(3.7.4)**,

$$\mathbf{i_1 = -7.5\ A,\ i_2 = -2.5\ A,\ i_3 = 3.93\ A,\ i_4 = 2.143\ A}$$

<u>Practice problem 3.7:</u> Use mesh analysis to determine i1, i2, and i3 in Figure shown below.

<u>Answer:</u> $i_1 = 3.474$ A, $i_2 = 0.4737$ A, $i_3 = 1.1052$ A.



3.5 NODAL ANALYSIS

Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables. Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously. To simplify matters, we shall assume in this section that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed in the next section. In nodal analysis, we are interested in finding the node voltages. Given a circuit with **n** nodes without voltage sources, the nodal analysis of the circuit involves taking the following three steps.

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages $\mathbf{v_1, v_2, . . v_{n-1}}$ to the remaining **n-1** nodes. The voltages are referenced with respect to the reference node.
2. Apply **KCL** to each of the **n-1** nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

We shall now explain and apply these three steps.

The first step in nodal analysis is selecting a node as the reference or datum node. The reference node is commonly called the ground since it is assumed to have zero potential. A reference node is indicated by any of the three symbols in **Fig. 3.1**. The type of ground in **Fig. 3.1(b)** is called a chassis ground and is used in devices where the case, enclosure, or chassis acts as a reference point for all circuits. When the potential of the earth is used as reference, we use the earth ground in **Fig. 3.1(a) or (c)**. We shall always use the symbol in **Fig. 3.1(b)**. Once we have selected a reference node, we assign voltage designations to nonreference nodes. Consider, for example, the circuit in **Fig. 3.2(a)**. Node 0 is the reference node ($\mathbf{v = 0}$), while nodes 1 and 2 are assigned voltages $\mathbf{v_1}$ and $\mathbf{v_2}$, respectively. Keep in mind that the node voltages are defined with respect to the reference node. As illustrated in **Fig. 3.2(a)**, each node voltage is the voltage rise from the reference node to the

corresponding nonreference node or simply the voltage of that node with respect to the reference node.

The number of nonreference nodes is equal to the number of independent equations that we will derive.

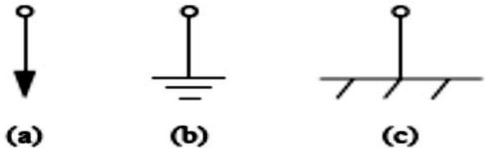


Figure 3.1 Common symbols for indicating a reference node.

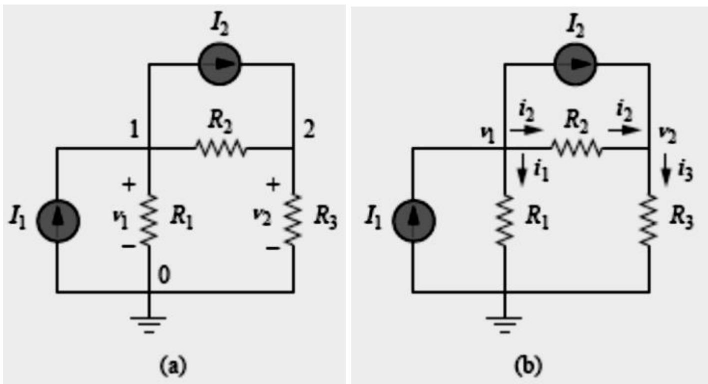


Figure 3.2 Typical circuits for nodal analysis.

As the second step, we apply **KCL** to each nonreference node in the circuit. To avoid putting too much information on the same circuit, the circuit in **Fig. 3.2(a)** is redrawn in **Fig. 3.2(b)**, where we now add $\mathbf{i_1}$, $\mathbf{i_2}$, and $\mathbf{i_3}$ as the currents through resistors $\mathbf{R_1}$, $\mathbf{R_2}$, and $\mathbf{R_3}$, respectively. At node 1, applying **KCL** gives

$$\mathbf{I_1 = I_2 + i_1 + i_2} \tag{3.1}$$

At node 2,

$$\mathbf{I_2 + i_2 = i_3} \tag{3.2}$$

We now apply Ohm's law to express the unknown currents $\mathbf{i_1}$, $\mathbf{i_2}$, and $\mathbf{i_3}$ in terms of node voltages. The key idea to bear in mind is that, since resistance is a passive element, by the passive sign convention, current must always flow from a higher potential to a lower potential.

Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as

$$i = \frac{v_{higher} - v_{lower}}{R} \tag{3.3}$$

Note that this principle is in agreement with the way we defined resistance in Chapter 2 (see **Fig. 2.3**). With this in mind, we obtain from **Fig. 3.2(b)**,

$$i_1 = \frac{v_1 - 0}{R_1}, \quad \text{or } i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2}, \quad \text{or } i_2 = G_2 (v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3}, \quad \text{or } i_3 = G_3 v_2 \tag{3.4}$$

Substituting **Eq. (3.4) in Eqs. (3.1)** and **(3.2)** results, respectively, in

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} \tag{3.5}$$

$$I_2 + v_1 - \frac{v_2}{R_2} = \frac{v_2}{R_3} \tag{3.6}$$

In terms of the conductances, Eqs. (3.5) and (3.6) become

$$I_1 = I_2 + G_1 v_1 + G_2 (v_1 - v_2) \tag{3.7}$$

$$I_2 + G_2 (v_1 - v_2) = G_3 v_2 \tag{3.8}$$

The third step in nodal analysis is to solve for the node voltages. If we apply **KCL** to **n−1** nonreference nodes, we obtain **n−1** simultaneous equations such as **Eqs. (3.5)** and **(3.6)** or **(3.7)** and **(3.8)**. For the circuit of **Fig. 3.2**, we solve **Eqs. (3.5)** and **(3.6)** or **(3.7)** and **(3.8)** to obtain the node voltages v_1 and v_2 using any standard method, such as the substitution method, the elimination method, Cramer's rule, or matrix inversion. To use either of the last two methods, one must cast the simultaneous equations in matrix form. For example, **Eqs. (3.7)** and **(3.8)** can be cast in matrix form as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \tag{3.9}$$

which can be solved to get v1 and v2. Equation 3.9 will be generalized in Section 3.6.

Example 3.1: Calculate the node voltages in the circuit shown in **Fig. 3.3(a)**.

Solution:

Consider **Fig. 3.3(b)**, where the circuit in **Fig. 3.3(a)** has been prepared for nodal analysis. Notice how the currents are selected for the application of **KCL**. Except for the branches with current sources, the labeling of the currents is arbitrary but consistent. (By consistent, we mean that if, for example, we assume that i2 enters the 4_resistor from the left-hand side, i2 must leave the resistor from the right-hand side.) The reference node is selected, and the node voltages v1 and v2 are now to be determined.

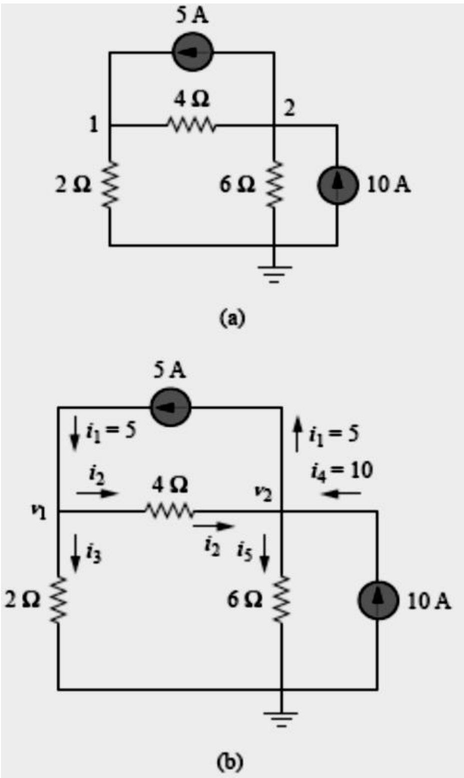


Figure 3.3 For Example 3.1: (a) original circuit, (b) circuit for analysis

At node 1, applying **KCL** and **Ohm's law** gives

$$\mathbf{i_1 = i_2 + i_3 \Rightarrow 5 = \frac{v1 - v2}{4} + \frac{v1 - 0}{2}}$$

Multiplying each term in the last equation by 4, we obtain

$$\mathbf{20 = v_1 - v_2 + 2v_1}$$

or

$$\mathbf{3v_1 - v_2 = 20} \qquad (3.1.1)$$

At node 2, we do the same thing and get

$$\mathbf{i_2 + i_4 = i_1 + i_5 \Rightarrow \frac{v1 - v2}{4} + 10 = 5 + \frac{v2 - 0}{6}}$$

Multiplying each term by 12 results in

$$3v_1 - 3v= + 120 = 60 + 2v_2$$

or

$$-3v_1 + 5v_2 = 60 \qquad (3.1.2)$$

Now we have two simultaneous **Eqs. (3.1.1)** and **(3.1.2)**. We can solve the equations using any method and obtain the values of v1 and v2.

METHOD 1: Using the elimination technique, we add **Eqs. (3.1.1)** and **(3.1.2)**.

$$4v_2 = 80 \Rightarrow v_2 = 20 \text{ V}$$

Substituting $v_2 = 20$ in Eq. (3.1.1) gives

$$3v_1 - 20 = 20 \Rightarrow v_1 = 40/3 = 13.33 \text{ V}$$

METHOD 2: To use **Cramer's rule**, we need to put **Eqs. (3.1.1)** and **(3.1.2)** in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix} \qquad (3.1.3)$$

The determinant of the matrix is

$$\Delta = D = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

We now obtain $\mathbf{v_1}$ and $\mathbf{v_2}$ as

$$\boldsymbol{v_1} = \frac{D_1}{D} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{D} = \frac{100+60}{12} = 13.33V$$

$$\boldsymbol{v_2} = \frac{D_2}{D} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{D} = \frac{180+60}{12} = 20V$$

giving us the same result as did the elimination method.

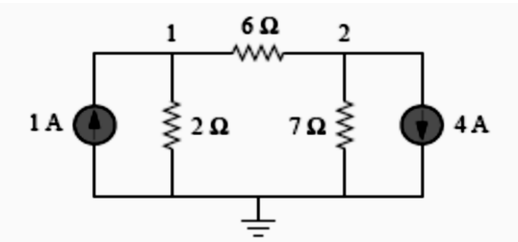
If we need the currents, we can easily calculate them from the values of the nodal voltages.

$$\mathbf{i_1 = 5\ A}, \qquad \boldsymbol{i_2} = \frac{\boldsymbol{v_1 - v_2}}{\mathbf{4}} = \mathbf{-1.6667\ A}, \qquad \boldsymbol{i_3} = \frac{\boldsymbol{v_1}}{\mathbf{2}} = \mathbf{6.666A},$$

$$\mathbf{i_4 = 10\ A}, \qquad \boldsymbol{i_5} = \frac{\boldsymbol{v_2}}{\mathbf{6}} = \mathbf{3.333A}$$

The fact that $\mathbf{i_2}$ is negative shows that the current flows in the direction opposite to the one assumed.

PRACTICE PROBLEM 3.1: Obtain the node voltages in the circuit in Figure below.
Answer: $v_1 = -2$ V, $v_2 = -14$ V.



Example 3.2: Determine the voltages at the nodes in **Fig. 3.4(a)**.

Solution:

The circuit in this example has three non reference nodes, unlike the previous example which has two non reference nodes. We assign voltages to the three nodes as shown in **Fig. 3.4(b)** and label the currents.

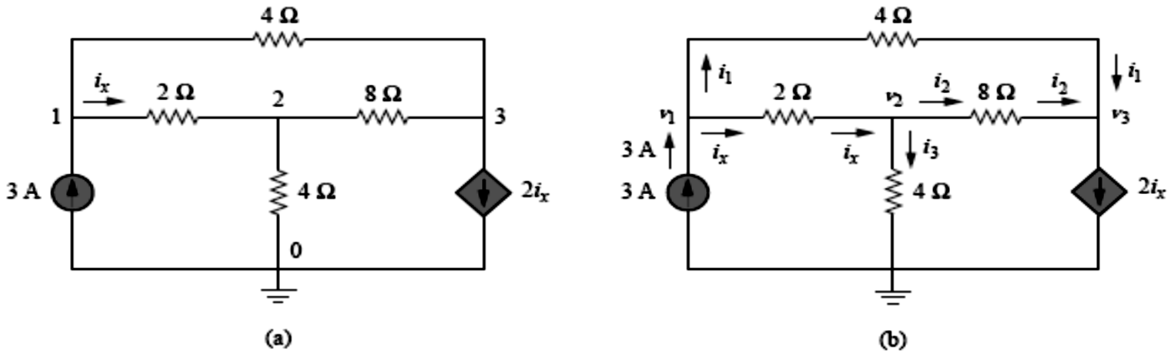


Figure 3.4 For Example 3.2: (a) original circuit, (b) circuit for analysis.

At node 1,

$$\mathbf{3 = i_1 + i_x \Rightarrow\ 3\ = \frac{v1 - v3}{4} + \frac{v1 - v2}{2}}$$

Multiplying by 4 and rearranging terms, we get

$$\mathbf{3v_1 - 2v_2 - v_3 = 12} \tag{3.2.1}$$

At node 2,

$$\mathbf{i_x = i_2 + i_3 \Rightarrow \frac{v1 - v2}{2} = \frac{v2 - v3}{8} + \frac{v2 - 0}{4}}$$

Multiplying by 8 and rearranging terms, we get

$$\mathbf{-4v_1 + 7v_2 - v_3 = 0} \tag{3.2.2}$$

At node 3,

$$\mathbf{i_1 + i_2 = 2i_x \Rightarrow \frac{v1 - v3}{4} + \frac{v2 - v3}{8} = \frac{2(v1 - v2)}{2}}$$

Multiplying by 8, rearranging terms, and dividing by 3, we get

$$\mathbf{2v_1 - 3v_2 + v_3 = 0} \tag{3.2.3}$$

We have three simultaneous equations to solve to get the node voltages **v_1, v_2**, and **v_3**. We shall solve the equations in **Cramer's rule** ways. We will put **Eqs. (3.2.1) to (3.2.3)** in matrix form.

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

From this, we obtain

$$\boldsymbol{v_1 = \frac{D_1}{D}}, \qquad \boldsymbol{v_2 = \frac{D_2}{D}}, \qquad \boldsymbol{v_3 = \frac{D_3}{D}}$$

where ***D***, ***D₁***, ***D₂***, and ***D₃*** are the determinants to be calculated as follows.

$$D = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = 21 - 12 + 4 + 14 - 9 - 8 = 10$$

Similarly, we obtain

$$D_1 = \begin{vmatrix} 12 & -2 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 1 \end{vmatrix} = 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$D_2 = \begin{vmatrix} 3 & 12 & -1 \\ -4 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$D_3 = \begin{vmatrix} 3 & -2 & 12 \\ -4 & 7 & 0 \\ 2 & -3 & 0 \end{vmatrix} = 0 + 144 + 0 - 168 - 0 - 0 = -24$$

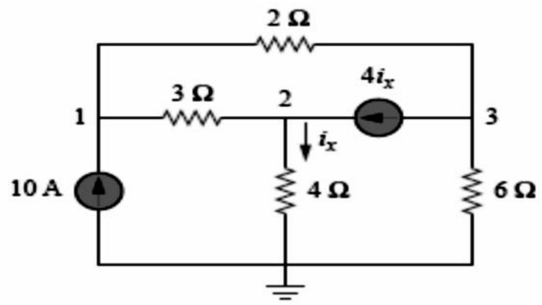
Thus, we find

$$\boldsymbol{v_1 = \frac{D_1}{D} = \frac{48}{10} = 4.8\,V}, \qquad \boldsymbol{v_2 = \frac{D_2}{D} = \frac{24}{10} = 2.4\,V}$$

$$\boldsymbol{v_3 = \frac{D_3}{D} = \frac{-24}{10} = -2.4\,V}$$

Practice problem 3.2: Find the voltages at the three nonreference nodes in the circuit of Figure below.

Answer: $v_1 = 80$ V, $v_2 = -64$ V, $v_3 = 156$ V.



3.6 NODAL ANALYSIS WITH VOLTAGE SOURCES

We now consider how voltage sources affect nodal analysis. We use the circuit in **Fig. 3.5** for illustration. Consider the following two possibilities.

CASE 1: If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source. In **Fig. 3.5**, for example,

$$\mathbf{v_1 = 10\ V} \tag{3.10}$$

Thus our analysis is somewhat simplified by this knowledge of the voltage at this node.

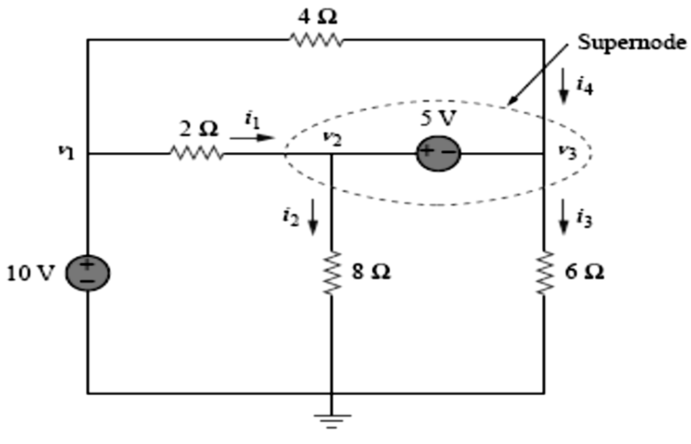


Figure 3.5 A circuit with a supernode.

CASE 2: If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a generalized node or supernode; we apply both **KCL** and **KVL** to determine the node voltages.

A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

In **Fig. 3.5**, nodes 2 and 3 form a supernode. (We could have more than two nodes forming a single supernode. For example, see the circuit in Fig. 3.14.) We analyze a circuit with supernodes using the same three steps mentioned in the previous section except that the supernodes are treated differently. Why? Because an essential component of nodal analysis is applying **KCL**, which requires knowing the current through each element. There is no way of knowing the current through a voltage source in advance. However, **KCL** must be satisfied at a supernode like any other node. Hence, at the supernode in **Fig. 3.5**,

$$\mathbf{i_1 + i_4 = i_2 + i_3} \quad (3.11a)$$

or

$$\frac{v1 - v2}{2} + \frac{v1 - v3}{4} = \frac{v2 - 0}{8} + \frac{v3 - 0}{6} \quad (3.11b)$$

To apply Kirchhoff's voltage law to the supernode in **Fig. 3.5**, we redraw the circuit as shown in **Fig. 3.6**. Going around the loop in the clockwise direction gives

$$\mathbf{-v_2 + 5 + v_3 = 0 \Rightarrow v_2 - v_3 = 5} \quad (3.12)$$

From **Eqs. (3.10), (3.11b),** and **(3.12)**, we obtain the node voltages.

Note the following properties of a supernode:

1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both **KCL** and **KVL**.

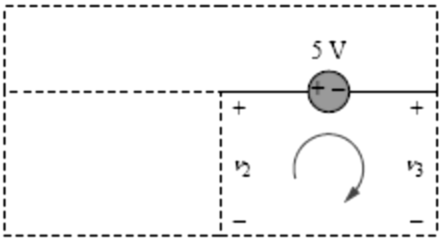


Figure 3.6 Applying KVL to a supernode.

Example 3.3: For the circuit shown in **Fig. 3.7**, find the node voltages.

Solution:

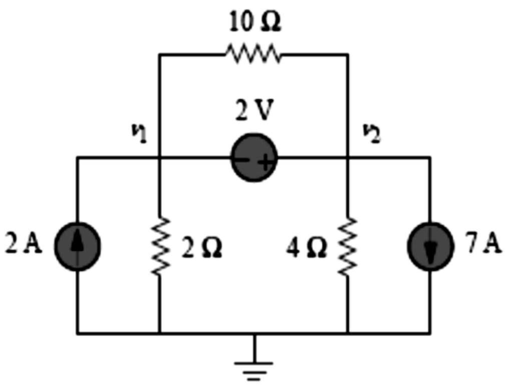


Figure 3.7 For Example 3.3.

The supernode contains the 2-V source, nodes 1 and 2, and the 10-Ω resistor. Applying **KCL** to the supernode as shown in **Fig. 3.8(a)** gives

$$\mathbf{2 = i_1 + i_2 + 7}$$

Expressing $\mathbf{i_1}$ and $\mathbf{i_2}$ in terms of the node voltages

$$\mathbf{2 = \frac{v1 - 0}{2} + \frac{v2 - 0}{4} + 7}$$

or

$$\mathbf{v_2 = -20 - 2v_1} \tag{3.3.1}$$

To get the relationship between $\mathbf{v_1}$ and $\mathbf{v_2}$, we apply **KVL** to the circuit in **Fig. 3.8(b)**. Going around the loop, we obtain

$$\mathbf{-v_1 - 2 + v_2 = 0 \;\Rightarrow\; v_2 = v_1 + 2} \tag{3.3.2}$$

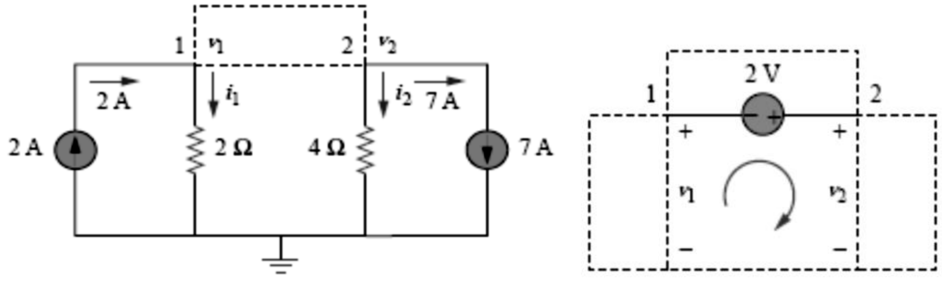
From **Eqs. (3.3.1)** and **(3.3.2)**, we write

$$\mathbf{v_2 = v_1 + 2 = -20 - 2v_1}$$

or

$$\mathbf{3v_1 = -22 \;\Rightarrow\; v_1 = -7.333\ V}$$

and $\mathbf{v_2 = v_1 + 2 = -5.333\ V}$. Note that the 10-Ω resistor does not make any difference because it is connected across the supernode.

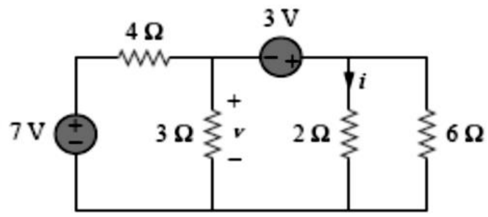


(a) **(b)**

Figure 3.8 Applying: (a) KCL to the supernode, (b) KVL to the loop.

Practice problem 3.3: Find v and **i** in the circuit in Figure below.

Answer: −0.2 V, 1.4 A.



Example 3.4: Find the node voltages in the circuit of **Fig. 3.9**.

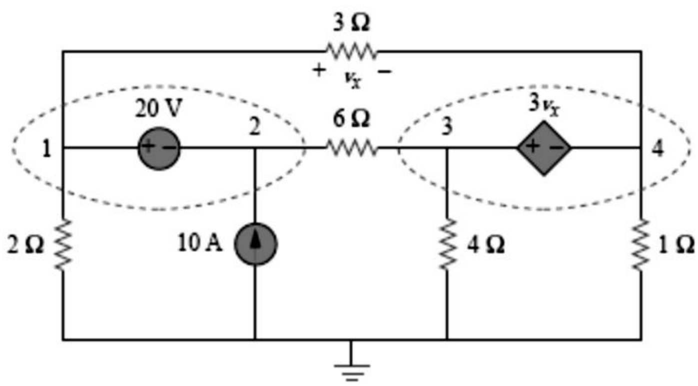


Figure 3.9 For Example 3.4.

Solution:

Nodes 1 and 2 form a supernode; so do nodes 3 and 4. We apply **KCL** to the two supernodes as in **Fig. 3.10(a)**. At supernode 1-2,

$$\mathbf{i_3 + 10 = i_1 + i_2}$$

Expressing this in terms of the node voltages,

$$\frac{v3 - v2}{6} + 10 = \frac{v1 - v4}{3} + \frac{v1}{2}$$

or

$$\mathbf{5v_1 + v_2 - v_3 - 2v_4 = 60} \qquad (3.4.1)$$

At supernode 3-4,

$$\mathbf{i_1 = i_3 + i_4 + i_5} \Rightarrow \quad \frac{v1 - v4}{3} = \frac{v3 - v2}{6} + \frac{v4}{1} + \frac{v3}{4}$$

or

$$\mathbf{4v_1 + 2v_2 - 5v_3 - 16v_4 = 0} \qquad (3.4.2)$$

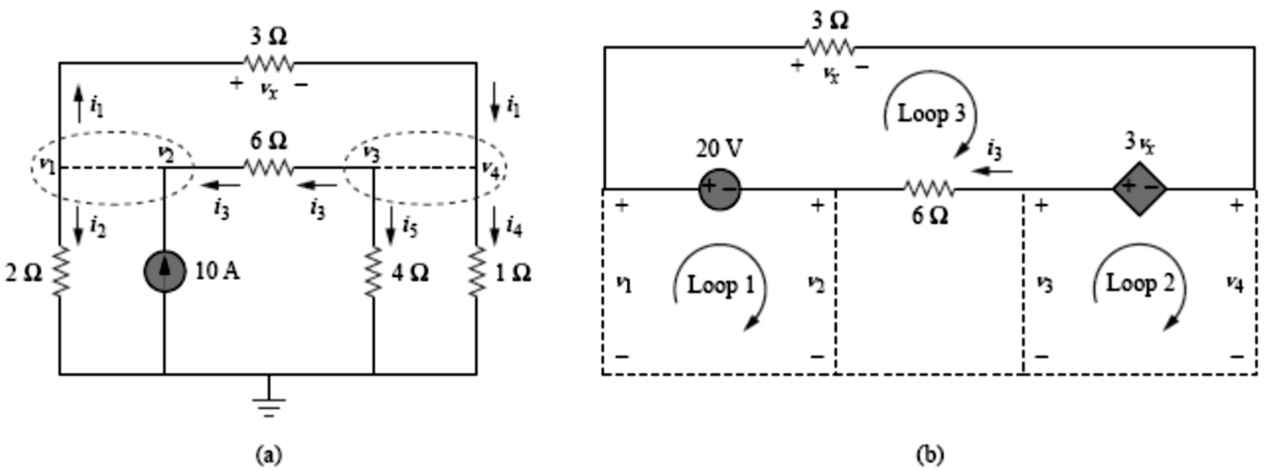


Figure 3.10 Applying: (a) KCL to the two supernodes, (b) KVL to the loops.

We now apply **KVL** to the branches involving the voltage sources as shown in **Fig. 3.10(b)**.

For loop 1,

$$\mathbf{-v_1 + 20 + v_2 = 0 \quad \Rightarrow \quad v_1 - v_2 = 20} \qquad (3.4.3)$$

For loop 2,

$$\mathbf{-v_3 + 3v_x + v_4 = 0}$$

But $\mathbf{v_x = v_1 - v_4}$ so that

$$\mathbf{3v_1 - v_3 - 2v_4 = 0} \qquad (3.4.4)$$

For loop 3,

$$\mathbf{v_x - 3v_x + 6i_3 - 20 = 0}$$

But $\mathbf{6i_3 = v_3 - v_2}$ and $\mathbf{v_x = v_1 - v_4}$. Hence

$$\mathbf{-2v_1 - v_2 + v_3 + 2v_4 = 20} \qquad (3.4.5)$$

We need four node voltages, $\mathbf{v_1, v_2, v_3}$, and $\mathbf{v_4}$, and it requires only four out of the five **Eqs. (3.4.1)** to **(3.4.5)** to find them. Although the fifth equation is redundant, it can be used to check results. We can eliminate one node voltage so that we solve three simultaneous equations instead of four. From **Eq. (3.4.3)**, $\mathbf{v_2 = v_1 - 20}$. Substituting this into **Eqs. (3.4.1)** and **(3.4.2)**, respectively, gives

$$\mathbf{6v_1 - v_3 - 2v_4 = 80} \qquad (3.4.6)$$

and

$$\mathbf{6v_1 - 5v_3 - 16v_4 = 40} \qquad (3.4.7)$$

Equations (3.4.4), (3.4.6), and (3.4.7) can be cast in matrix form as

$$\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{6} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{6} & \mathbf{-5} & \mathbf{-16} \end{bmatrix} \begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{80} \\ \mathbf{40} \end{bmatrix}$$

Using Cramer's rule,

$$D = \begin{vmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{6} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{6} & \mathbf{-5} & \mathbf{-16} \end{vmatrix} = -18, \qquad D_1 = \begin{vmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{80} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{40} & \mathbf{-5} & \mathbf{-16} \end{vmatrix} = -480$$

$$D_3 = \begin{vmatrix} \mathbf{3} & \mathbf{0} & \mathbf{-2} \\ \mathbf{6} & \mathbf{80} & \mathbf{-2} \\ \mathbf{6} & \mathbf{40} & \mathbf{-16} \end{vmatrix} = -3120, \qquad D_4 = \begin{vmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{0} \\ \mathbf{6} & \mathbf{-1} & \mathbf{80} \\ \mathbf{6} & \mathbf{-5} & \mathbf{40} \end{vmatrix} = 840$$

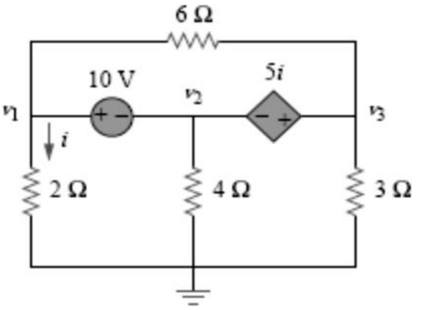
Thus, we arrive at the node voltages as

$$\boldsymbol{v_1} = \frac{\boldsymbol{D_1}}{\boldsymbol{D}} = \frac{\mathbf{-480}}{\mathbf{-18}} = \mathbf{26.667}\ \boldsymbol{V}, \qquad \boldsymbol{v_3} = \frac{\boldsymbol{D_3}}{\boldsymbol{D}} = \frac{\mathbf{-3120}}{\mathbf{-18}} = \mathbf{173.333}\ \boldsymbol{V}$$

$$\boldsymbol{v_4} = \frac{\boldsymbol{D_4}}{\boldsymbol{D}} = \frac{\mathbf{840}}{\mathbf{-18}} = \mathbf{-46.667}\ \boldsymbol{V}$$

and $\mathbf{v_2 = v_1 - 20 = 6.667\ V}$. We have not used **Eq. (3.4.5)**; it can be used to cross check results.

Practice problem 3.4: Find $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ in the circuit in Figure below using nodal analysis.



Answer: $v_1 = 3.043$ V, $v_2 = -6.956$ V, $v_3 = 0.6522$ V.